Integrating M
into the High Sc

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Textbook word problems can be modified to change the way the questions are posed without changing the prerequisite content.

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Analysis from the Third International Mathematics and Science Study led to the conclusion that mathematics education in the United States is "a mile wide and an inch deep" (Schmidt, McKnight, and Raizen 2002). According to a 2012 study from the Program for International Student Assessment, twenty-nine out of sixty-five participating nations and other jurisdictions outperformed the United States in mathematics by a statistically significant margin (up from twenty-three in 2009) (Heitin 2013). To improve our students' mathematical understanding, we need to focus on fewer topics but in greater detail. To accomplish this, the Common Core State Standards (CCSSI 2010) highlights specific mathematical skills and concepts with a three-step approach: focus, coherence, and rigor. The inclusion of modeling, as a Standard for Mathematical Practice and also as a content category, demands a more rigorous understanding of mathematical tools and how they can be used in everyday situations.

CCSSM's refocusing requires an overhaul with new books and other materials geared toward this new approach. Adapting the systems currently in place will take time. To ease the transition for in-service teachers, this article provides suggestions on how to incorporate modeling into the classroom with the resources already available.

WHAT IS MATHEMATICAL MODELING?

A mathematical model is a representation of a system (incorporating equations, functions, graphs, or other mathematical tools) that is used to help understand, explain, or predict the behavior of the system. Models can be simple—for example, using average speed and time to determine distance traveled—or complex—for example, using differential equations to predict how several competing species will interact.

Mathematical modeling is the process of using the mathematical tools at hand to describe the world around us. In the most general sense, modeling involves identifying a question, posing it

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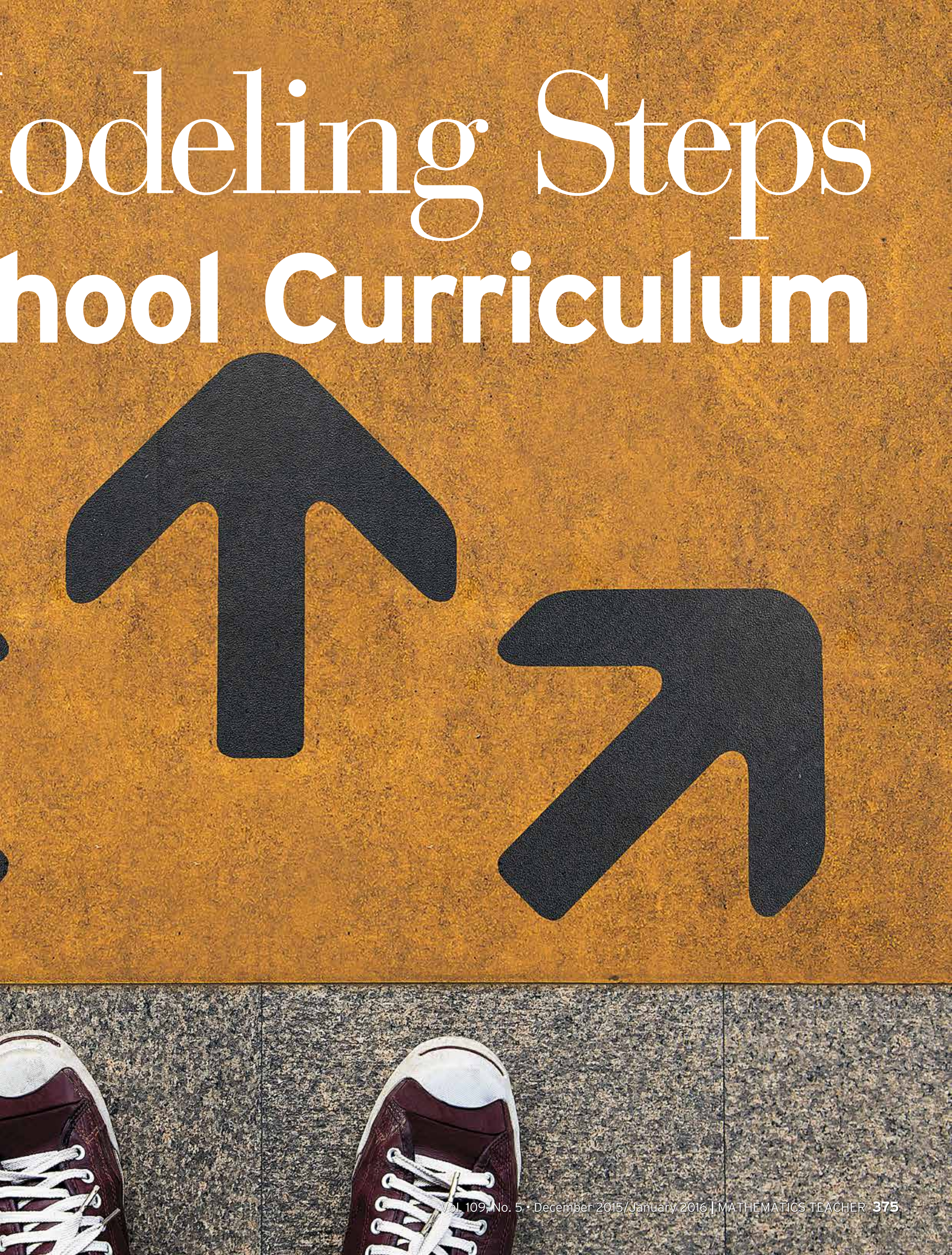

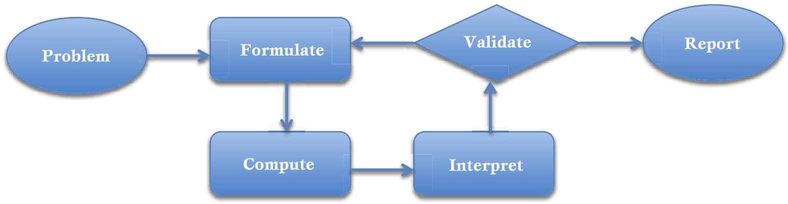

1. Problem: What is our question? What do we want to model, and what are the essential factors?
2. Formulate: How do we choose to model it? What representation is appropriate?
3. Compute: What does the model tell us?
4. Interpret: What does it mean?
5. Validate: Does it make sense? If not, reformulate a different model.
6. Report: Can we explain our results to others?

Source: Adapted from CCSSI (2010, p. 72)

Fig. 1 The basic modeling cycle, as shown in the Common Core State Standards for Mathematics, has six steps (CCSSI 2010, p. 72).

in a mathematical framework, solving the resulting problem, and then interpreting the results in the context of the original question. The modeling process as described in CCSSM is depicted in **figure 1**; variations can be found in (Meerschaert 2007) and (Blomhoj and Jensen 2003), among many others.

Word problems, the closest contact that many students have had to modeling, can be effective in some contexts, but most fall solely in the "compute" category. Only a select few delve into the "interpret" stage, giving students very little experience with the other (perhaps more important) parts of the modeling process. The new emphasis on modeling leaves educators with a difficult choice: Take extra class time for students to complete an entire modeling cycle, require students to complete part or all the modeling process outside class, or have students complete smaller tasks while still covering the whole spectrum of modeling.

MODELING PROJECTS

For many current high school students, any introduction to the modeling process has generally been in the form of a project or activity spanning multiple several class periods and interpreted as a break from "real" mathematics. Although modeling projects are excellent ways to get students excited about mathematics and show them some real-world applications, they are also quite time-consuming: worthwhile activities must be developed, a number of class periods must be devoted to following the project through to completion and the assessment of student work.

Modeling activities such as those found in Gould, Murray, and Sanfratello (2012) or Bliss, Fowler, and Galluzzo (2014) are great starting points to get ideas of the types of modeling activities that teachers use in their classrooms. However, activities such as these may take students longer or be less meaningful if they appear to be coming out of nowhere. For modeling to have a lasting impact on students, a modeling mentality should be developed throughout the mathematics curriculum, not just once or twice a year.

ADAPTING TEXTBOOK PROBLEMS

To help build a modeling mentality, we will isolate some common types of problems from popular high school mathematics textbooks and provide suggestions for modifying these problems to highlight specific components of modeling. Take, for example, a typical problem from an Algebra 1 textbook (A. E. Bellman et al. 2009):

Sample problem: Suppose your group recorded a CD. Now you want to copy and sell it. One company charges \$250 for making a master CD and designing the art for the cover. There is also a cost of \$3 to burn each CD. The total cost, $P(c)$, depends on the number of CDs (c) burned. Use the function rule, $P(c) = 250 + 3c$, to make a table of values and a graph.

The emphasis of this problem is to practice using function notation. Enough information is provided to understand why the function rule as given is appropriate, but many students will simply skim over the body of the problem and focus on the last sentence to generate their solution. Like many similar word problems, this problem centers on the "compute" step of the modeling process. Although function notation and the ability to use a model to find specific values are important skills, the "formulate" and "interpret" steps are virtually ignored in current texts.

This sample problem does not really differ from a purely computational problem such as this: Graph the function $f(x) = 250 + 3x$ by first listing a table of values. Here, the real-world situation serves simply as a pretense and exemplifies Pollak's quip that "the purpose of a word problem is only to practice the mathematics of the current chapter" (Pollak 2011). This problem is not a modeling problem as written, but by highlighting some of the structure that is already laid out in the problem or by modifying the focus, this type of problem can serve as an excellent introduction to mathematical modeling.

Some possible modifications to the sample problem follow:

Restatement 1: Suppose that you are interested in starting a CD production company. Your clients provide the music and artwork, and your company produces the CDs for distribution. What factors should go into the price that you charge your customers?

Discussion questions: Should you charge someone who wants to produce two CDs twice as much as you would charge someone who wants to produce only one? What if that person wants 10,000 CDs?

Asking leading questions such as these allows teachers to cater the lesson to the learning objectives for the activity. In this instance, the goal is to shift the focus from the "compute" step to the "problem" step. This revision does not even ask students to provide mathematical symbols, only for them to think of the most important components of a price structure. Depending on students' responses, this revision could easily lead to the formulation of a model similar to the one provided in the sample problem, or students may choose additional complexity (e.g., they might charge an hourly rate for the time to create the CDs as well as the materials used).

The importance of an exercise like this is not for students to get a correct answer and promptly forget about it but for them to see functions as tools that allow us to quickly find appropriate values based on the most essential factors (in this case, pricing based on the number of CDs or the amounts of materials used). In addition, any discussion about the different models that were chosen can lead into a discussion about the "validate" step for refining a model. Although few artists specifically create music for CD sales today, we may instead decide to build a problem around any activity with a fixed cost per item and a startup cost to show the practicality of linear models.

A similar type of question appears in an Algebra 2 problem (from A. E. Bellman et al. 2004), which asks students to compare the costs of using a cable service for $29.95 per month with renting videos at a cost of $2.95 each. This problem can also be modified like restatement 1 by asking students to come up with the costs that are associated with a group of entertainment providers and develop functions that represent the costs in relation to usage. Very few textbook problems ask students to consider what actually goes into creating a model, so rather than modifying given problems, we might consider creating problems from topics pertinent to student interests.

Very few textbook problems ask students to consider what actually goes into creating a model.

The next restatement asks students to formulate a model that fits provided data:

Restatement 2: Suppose that your group recorded a CD. Now you want to copy and sell it. One company charges $250 for making a master CD and designing the cover art. It charges $400 to produce 50 CDs and $550 to produce 100 CDs.

- Assuming that the company charges the same amount for each additional CD, predict the cost of producing 600 CDs.
- Create a linear function that represents how much it would cost to produce *c* CDs.
- What are the units for the number in front of the *c* in your function?

In addition to formulating the model, students are asked to think about the significance of the parameters by revisiting the function and recognizing the fixed overhead ($250 for the master CD and artwork) and the price per CD ($3). This information was provided in the original problem, but removing the function and having students reason out the significance for themselves makes the problem much more meaningful. Encountering a problem with just a few data points will also help prepare students for the future when they want to determine the appropriate regression model to use with larger data sets.

A common structure throughout typical high school mathematics textbooks is giving a general rule or formula—for example, $y = ab^x$, $A = P(1 + r/n)^{nt}$— followed by a multitude of examples to practice substituting numbers in the appropriate locations. Unfortunately, this approach reinforces the mindset that solving word problems is just about finding the appropriate place to stick the provided numbers. Textbooks are loaded with examples such as this one: "A computer valued at $6500 depreciates at a rate of 14.3% per year. Write a function that models the value of the computer" (A. E. Bellman et al. 2004). In this

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chapter of the textbook, the formula $y = ab^x$ is introduced as a general exponential function, and this problem asks students to recognize that if they substitute $a = 6500$ and $b = 0.857$ into this function, they will get the correct result. It would be more meaningful to introduce $y = ab^x$ as an exponential model that represents all forms of exponential growth and decay, provide data points, and ask students to find the parameter values and for a specific model.

Here is one reworking: "A computer originally valued at \$6500.00 has depreciated in value to \$5570.50 by the end of the year. Create a model that represents the value of the computer x years from its purchase date."

Better yet, here's another reworking:

Restatement 3: Suppose that your group recorded a CD and you want to make professional copies for sale. A local company charges \$250 for making a master CD and designing the cover art. It quotes a price of \$400 to produce 50 CDs. Two price functions for producing CDs,

$$P(c) = 250 + 3c \text{ and } Q(c) = 250 - 47c + c^2,$$

both meet the quoted prices for making zero CDs and for making 50. Which is a better model and why?

Although a different approach could emphasize creating the model, this problem is specifically designed to engage students in the "interpret" and "validate" steps of the modeling process. The ability to step back from a problem and question whether the result makes sense is an essential skill. Every time students are introduced to new models, it is worthwhile to give them a list of events and ask them what the appropriate model form would be: linear (cost when buying multiple hamburgers, distance over a given time, etc.); exponential (automobile depreciation, spread of a disease over time, etc.); quadratic (height of a falling body over time, area of a square in relation to side length, etc.); and so on.

One final example gives a reason why we should care about creating models in the first place:

Restatement 4: Suppose that your group recorded a CD and you want to make professional copies for sale. One company charges \$250 for making a master CD and designing the cover art and \$3 to burn each CD. Another company charges \$300 for making a master CD and designing the cover and \$2.50 to burn each CD.

- What criterion should you use when deciding which company to choose?
- At what point would you consider changing companies?

This is a prototypical break-even analysis question that is slightly modified from a follow-up to the original sample problem (Bellman et al. 2009). Students are asked to formulate two similar models and use them to make an informed decision between competing offers on the basis of their personal habits. These types of problems generally start appearing in Algebra 2 textbooks but can certainly be introduced earlier to reinforce practical advantages of a modeling mentality.

AN INTEGRATED, INCREMENTAL PROCESS

The explicit importance of mathematical modeling as part of the standards requires a paradigm shift in how mathematics is taught. Students at all levels need experience working with, interpreting, and analyzing existing models as well as formulating new models. The process outlined in **figure 1** provides an excellent framework for constructing exercises and activities to train students in the art of mathematical modeling; this article provides suggestions and resources for building on that foundation.

Modeling projects can be exceptional learning tools, but many instructors feel that introducing mathematical modeling opportunities would require them to eliminate important content matter. After all, adding two days of mathematical modeling means subtracting two days of something else. When we tack on a modeling project to the end of a semester or occasionally assign a handful of contrived word problems, mathematical modeling comes off as a diversion. For students to become proficient modelers, they need to experience pieces of the modeling process sprinkled throughout their mathematical careers.

Like many skills, modeling is best developed through practice of smaller components gradually accumulated over time. Using the techniques outlined in this article—identifying pertinent factors (restatement 1), interpreting model parameters (restatement 2), comparing and critiquing model types (restatement 3), and using models to make informed decisions (restatement 4)—teachers can modify existing examples, exercises, and activities to focus on each of the different aspects of the mathematical modeling process. In this way, mathematical modeling becomes part of the daily mathematical instruction.

BIBLIOGRAPHY

Bellman, Allan E., Sadie Chavis Bragg, Randall I. Charles, Basia Hall, William G. Handlin Sr., and Dan Kennedy. 2009. *Algebra I.* Boston: Pearson Prentice Hall.

Bellman, Allen E., Sadie C. Bragg, Randall I. Charles, William G. Handlin, and Dan Kennedy. 2004. *Algebra 2.* Upper Saddle River, NJ: Prentice Hall.

Bliss, Karen M., Kathleen R. Fowler, and Benjamin J. Galluzzo. 2014. *Math Modeling: Getting Started and Getting Solutions.* Philadelphia: Society for Industrial and Applied Mathematics.

Blomhoj, Morten, and Tomas Hojgaard Jensen. 2003. "Developing Mathematical Modelling Competence: Conceptual Clarification and Educational Planning." *Teaching Mathematics and Its Applications* 22 (3): 123–39.

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Gould, Heather, Diane R. Murray, and Andrew Sanfratello. 2012. *Mathematical Modeling Handbook.* Consortium for Mathematics and Its Applications. http://www.comap.com/modelingHB/Modeling_HB_Sample.pdf

Heitin, Liana. 2013. "U.S. Achievement Stalls as Other Nations Make Gains." *Education Week*, December 3.

Meerschaert, Mark. 2007. *Mathematical Modeling.* Burlington, MA: Elsevier.

Pollak, Henry O. 2011. "What Is Mathematical Modeling?" *Journal of Mathematics Education at Teachers College* 2 (1): 64.

Schmidt, William H., Curtis C. McKnight, and Senta A. Raizen. 2002. *A Splintered Vision: An Investigation of U.S. Science and Mathematics Education.* New York: Kluwer Academic Publishers.


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