COLLEGE IV

MATHEMATICS MODULE 57-36-25

CALCULUS AND ANALYTIC GEOMETRY III: PARTIAL DERIVATIVES

A study of partial derivatives of functions of two variables and their applications in finding differentials, directional derivatives, tangent planes, and relative extrema.

Ted Sundstrom

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MODULE SELECTION INFORMATION:

57-36-25

Partial Derivatives

1.0 Credit

MATHEMATICS

COLLEGE IV

Everything about the design of College IV is aimed at making access to education easier. There are no scheduled classes, so students may start whenever they wish, drop out for unspecified periods of time, and then return. The curriculum has been packaged into small units called modules. The individualized format of College IV puts students in a one-to-one relationship with each of their instructors. Since each student must master the assigned materials before moving on, the quality of the education is maintained at a high level. In addition to the self-paced learning modules, students do independent or group study through contracts which they develop with the faculty. The B.A. and B.S. degrees are offered, but students not seeking degrees are encouraged to study whatever subjects interest them.

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MATHEMATICS MODULE 57-36-25

PARTIAL DERIVATIVES

PREREQUISITES

Mathematics Module 57-36-20, "Solid Analytic Geometry"

MATERIALS NEEDED

The text, *Calculus and Analytic Geometry*, second edition, by Douglas F. Riddle and published by Wadsworth. In addition, you may find it useful to consult the Schaum College Outline Series book, *Calculus*, published by McGraw-Hill. This contains many problems with detailed solutions. Both books are available in the campus bookstore.

If you wish, also consult any calculus text by looking in the index under "Partial Derivatives" or "Functions of Several Variables." These other texts may often present the same material from a slightly different viewpoint.

LOCATION

Anywhere

RATIONALE

Differential calculus of a function of one variable is very powerful in solving a great variety of problems. However, many problems require the use of several independent variables, and thus it is necessary to devise methods for analyzing functions of several variables. In this module, you will study functions of two variables and examine how some concepts of differential calculus can be defined for these functions and then applied to help in the analysis of these functions.

OBJECTIVES

Terminal Objective

You will be able to determine the partial derivatives of a function of two variables and use them to find derivatives of implicit functions, total differentials, directional derivatives, tangent planes of surfaces, and the relative maxima and minima of a function of two variables.

Enabling Objectives

36-25.1 You will be able to (1) explain the concept of a limit of a function of two variables, and (2) evaluate certain limits of functions of two variables or demonstrate that the limit does not exist.

36-25.2 You will be able to determine the partial derivatives and higher order partial derivatives of a function of several variables.

36-25.3 You will be able to use the chain rule to determine derivatives and partial derivatives.

36-25.4 You will be able to use partial derivatives to determine derivatives and partial derivatives of implicit functions.

36-25.5 You will be able to determine the total differential of a function of two variables and to determine when a differential expression is an exact differential.

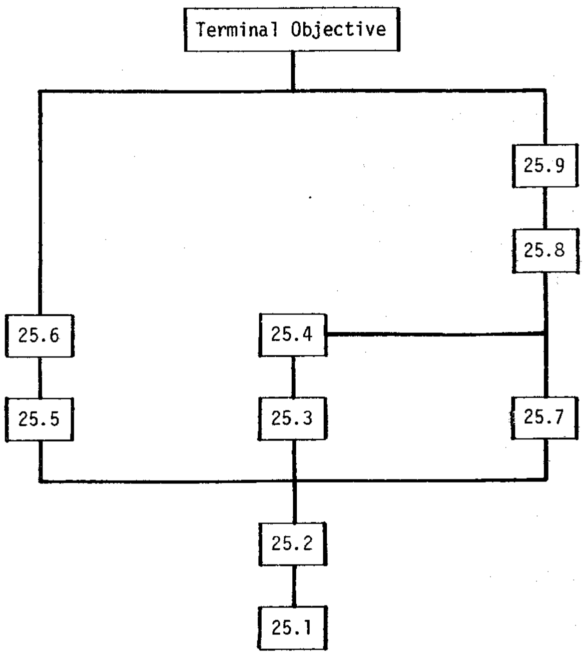
36-25.6 You will be able to use the total differential in approximation problems.

36-25.7 You will be able to find the gradient of a function of two variables and the directional derivatives of a function of two variables.

36-25.8 You will be able to find the equations of the tangent plane and the normal line of a surface at a given point.

36-25.9 You will be able to determine the relative maxima, relative minima, and saddle points of a function of two variables.

LEARNING HIERARCHY



LEARNING ACTIVITIES

Instructions

The module is divided into units corresponding to chapter sections in the text. Each of the units contains most of the following titles: Unit Objective, Premeasure, Study Pages, Learning Supplement, and a Self-Assessment test. These titles should be used and studied in the order in which they appear. For example, in some sections the study pages will occur before the explanatory material while in other sections this order might be reversed.

Upon completion of all the learning activities, you should take the Module Self-Assessment test found on the last page(s) of this pamphlet. This self-assessment test is just another form of the mastery exam that you will take for credit. Thus, if you can work 90% of the exercises on the self-assessment test correctly, you should also be able to successfully complete the mastery exam at the 90% level.

Remember that there are tutors and instructors in College IV who want to help you with *any* questions you have.

If you believe that you already understand the material presented in a unit, try the self-assessment for the unit first. If you *successfully* complete it, go directly to the next unit.

Each premeasure involves concepts necessary for successful completion of the unit. If you have trouble with a premeasure, contact a tutor or instructor.

If you have trouble with an assigned exercise, check the corresponding example which appears either in the text or in Schaum's book.

The answers to all exercises that appear in the module are in the "Answer Section" of the module under the same unit number.

Unit 36-25.1

36-25.1 You will be able to (1) explain the concept of a limit of a function of two variables, and (2) evaluate certain limits of functions of two variables or demonstrate that the limit does not exist.

Premeasure

Evaluate each of the following limits or show that it does not exist.

1. $\lim\limits_{x \to 0^+} \frac{|x|}{x}$ 2. $\lim\limits_{x \to 0^-} \frac{|x|}{x}$ 3. $\lim\limits_{x \to 0} \frac{|x|}{x}$

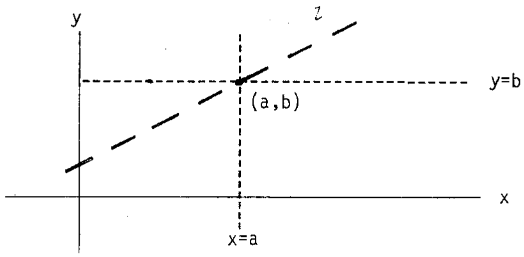
Study Pages

Riddle, pages 668-670, Section 21.1

Learning Supplement

It is helpful to compare, in intuitive terms, the definition on page 668 with that for $\lim\limits_{x \to a} f(x) = L$. Intuitively, $\lim\limits_{x \to a} f(x) = L$ means: (1) There exist numbers in the domain of f arbitrarily close to a, and (2) the distance from f(x) to L (i.e., $|f(x) - L|$) approaches zero as x gets closer to a. This is very similar to what is said in the definition on page 668 for $\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y) = L$. Part (a) means that there exist points in the domain of f arbitrarily close to (a,b), and (b) means that the distance from f(x,y) to L approaches zero as (x,y) approaches (a,b) from any direction.

Note the importance of the last part that states, "as (x,y) approaches (a,b) from any direction." For functions of one variable, x can approach a in only two directions, from the left or from the right; that is, $\lim\limits_{x \to a} f(x)$ exists if $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$. However, for functions of two variables, (x,y) can approach (a,b) in "infinitely" many ways. One technique used to approach (a,b) from various directions is to approach (a,b) along straight lines through (a,b). For example, you could determine:



1. $\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y)$ as (x,y) approaches (a,b) along the line $y=b$

2. $\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y)$ as (x,y) approaches (a,b) along the line $x=a$

3. $\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y)$ as (x,y) approaches (a,b) along the line l. You must determine the equation of l.

When you restrict (x,y) to being on a straight line, you should be able to express $z = f(x,y)$ as a function of one variable. For example if $z = f(x,y) = x^2y - 3xy$, then,

a. Along the line $y = 2$, $z = f(x,2)$, $= 2x^2 - 6x$

b. Along the line $x = 1$, $z = f(1,y) = y - 3y = -2y$

c. Along the line $y = x + 1$, $z = f(x,x+1) = x^2(x+1) - 3x(x+1)$

Getting back to limits, in order for $\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y)$ to exist, the same value for the limit must be obtained no matter how (x,y) approaches (a,b). Study Example 2, page 670. Here it was shown that the limit does not exist by showing that different values for the limit are obtained when (x,y) approaches (0,0) along the line y = 0 and when (x,y) approaches (0,0) along the line x=0. Sometimes lines that are not parallel to the coordinate axes must be used. This is shown in the following example.

Example: Show that $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sin(xy)}{x^2+y^2}$ does not exist.

Solution: As (x,y) approaches (0,0) along the line y=0, $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sin(xy)}{x^2+y^2} =$ $\lim_{x \to 0} \frac{0}{x^2} = 0$. Similarly, as (x,y) approaches (0,0) along the line x=0, $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sin(xy)}{x^2+y^2} = 0$. However, as (x,y) approaches (0,0) along the line y=x,

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sin(xy)}{x^2+y^2} = \lim_{x \to 0} \frac{\sin x^2}{2x^2} = \lim_{x \to 0} \frac{2x\cos x^2}{4x} =$$

$$\lim_{x \to 0} \frac{2\cos x^2 - 4x^2\sin x^2}{4} = \frac{2}{4} = \frac{1}{2}.$$

Thus, $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sin(xy)}{x^2+y^2}$ does not exist.

<u>Exercises</u>

Riddle, page 674, problems 31, 32, 33

<u>Self-Assessment 36-25.1</u>

1. Show that $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.
2. Evaluate the limit of Problem 1 if (0,0) is approached along the line, y = 2x.
3. Show that $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{e^x \sin y}{x+y}$ does not exist.

Unit 36-25.2

> 36-25.2 You will be able to determine the partial derivatives and higher order partial derivatives of a function of several variables.

Study Pages

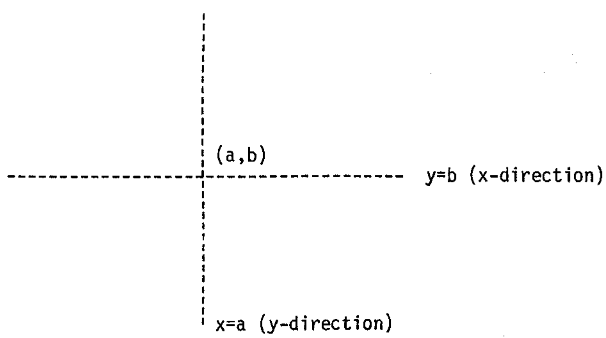
Riddle, pages 670-673, Section 21.2

Learning Supplement

Study the definition of $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ on pages 670-671. There are two very important things to keep in mind when dealing with partial derivatives.

(1) When finding $\frac{\partial f}{\partial x}$ (or $\frac{\partial f}{\partial y}$), keep in mind that y (x) is to be treated as a constant when differentiating. For example, if $z = f(x,y) = ye^x - x^2y$, then $\frac{\partial f}{\partial x} = ye^x - 2xy$. This is "just like" finding $g'(x)$ for $g(x) = 2e^x - 3x^2 \left[g'(x)=2e^x-3\cdot(2x)\right]$.

(2) Geometrically, for $z = f(x,y)$, $\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$ (= the partial derivative of f wrt x at (a,b)) is the slope at (a,b,f(a,b)) of the curve formed by the intersection of the surface of $z = f(x,y)$ and the vertical plane y=b, i.e., (x,y) is restricted to the line y=b, the x-direction.



The equation of this curve is $z = g(x) \doteq f(x,b)$, and its slope at $(a,b,f(a,b))$ is $g'(a) = \frac{\partial f}{\partial x}\Big|_{(a,b)}$.

Similarly, $\frac{\partial f}{\partial y}\Big|_{(a,b)}$ is the slope of $z = f(x,y)$ at $(a,b,f(a,b))$ in the y-direction, i.e., the slope at $(a,b,f(a,b))$ of the curve formed by the intersection of the surface of $z = f(x,y)$ and the vertical plane $x=a$.

Example: If $z = f(x,y) = ye^x + \frac{y^2}{x}$, then

$$\frac{\partial f}{\partial x} = ye^x - \frac{y^2}{x^2}, \quad \frac{\partial f}{\partial y} = e^x + \frac{2y}{x}, \text{ and } \frac{\partial f}{\partial x}\Big|_{(2,1)} = e^2 - \frac{1}{4},$$

$$\frac{\partial f}{\partial y}\Big|_{(2,1)} = e^2 + 1.$$

Although this is the best way to find partials, it might be helpful to consider this by looking at the intersection of the surface represented by $z = f(x,y) = ye^x + \frac{y}{x}$ and the plane $y=1$. Then $z = g(x) = f(x,1) = e^x + \frac{1}{x}$, and $g'(x) = e^x - \frac{1}{x^2}$. Thus, $g'(2) = e^2 - \frac{1}{4} = \frac{\partial f}{\partial x}\Big|_{(2,1)}$.

As a final note, you should be sure you understand the notation for second partial derivatives. In the above example,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(ye^x - \frac{y^2}{x^2}\right) = ye^x + \frac{2y^2}{x^3}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(e^x + \frac{2y}{x}\right) = \frac{2}{x}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(ye^x - \frac{y^2}{x^2}\right) = e^x - \frac{2y}{x^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(e^x + \frac{2y}{x}\right) = e^x - \frac{2y}{x^2}$$

Exercises

Riddle, pages 673-674, Section 21.2, Problems 1-27 (odd)

Self-Assessment 36-25.2

For Problems 1 and 2, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ at the point indicated.

1. $z = xy\cos y^2$ at (1,0,1)

2. $z = xe^{\frac{x}{y^2}}$ at (0,1,1)

For problems 3 and 4, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial y \partial x}$, $\frac{\partial^2 z}{\partial x \partial y}$

3. $z = \frac{x\cos y}{y^2}$

4. $z = \ln(x^2 + y^2)$

Unit 36-25.3

36-25.3	You will be able to use the chain rule to determine derivatives and partial derivatives.

Premeasure

Find $\frac{dy}{dx}$ for:

1. $y = \sin(4x^2+2x)$
2. $y = e^u$, $u = \sin x^2$

Study Pages

Riddle, pages 674-679, Section 21.3

Learning Supplement

If you are wondering which theorem in this section is called the chain rule, the answer is that they all are. In determining which theorem to use for a particular function and whether to use an ordinary derivative or a partial derivative, you should first ask yourself, "Is the function in question a function on one or more variables?" The (correct) answer to this question should help you. For example:

(1) In Theorem 21.2, z is a function of x and y, (so $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ must be used), and x,y are functions of t, (so $\frac{dx}{dt}$, $\frac{dy}{dt}$ must be used). Hence, we can consider z a function of t and compute $\frac{dz}{dt}$ by

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

(2) In Theorem 21.3, z is a function of x and y, and y is a function of x. This is a special case of Theorem 21.2 with t=x ($\frac{dx}{dt} = \frac{dx}{dx} = 1$), and so,

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}.$$

Note the distinction between $\frac{\partial z}{\partial x}$ and $\frac{dz}{dx}$ in this formula. This is explained on the bottom of page 677 in Riddle.

(3) In Theorem 21.5, z is a function of x,y (so $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ must be used, and x,y are both functions of u,v (so $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial u}$, $\frac{\partial y}{\partial u}$, $\frac{\partial y}{\partial v}$ must be used). Thus, z is a function of u and v and $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial u}$ can be calculated using the formulas in Theorem 21.5

You should now go back and restudy the examples in the text keeping these ideas in mind.

Example: If $z = \frac{x}{y}$ and x = usinv, y = ucosv, find $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$

Solution: z is a function of x,y and x,y are functions of u,v. Thus, z is a function of u,v and $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$ make sense.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{1}{y} \cdot \text{sinv} + \frac{-x}{y^2} \cdot \text{cosv} =$$

$$\frac{y\,\text{sinv} - x\,\text{cosv}}{y^2}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{1}{y} \cdot u\,\text{cosv} - \frac{x}{y^2} \cdot (-u\,\text{sinv}) =$$

$$\frac{yu\,\text{cosv} + ux\,\text{sinv}}{y^2}$$

Exercises

Riddle, page 679-680, Section 21.3, Problems 1,3,7,9,11,12,21,23,30,31

Solutions: 12) $\frac{\partial z}{\partial u} = 4(2u+v)(2v-u) - (2u+v)^2$

$\frac{\partial z}{\partial v} = 2(2u+v)(2v-u) + 2(2u+v)^2$

30) $\frac{dv}{dt} = 65\pi \frac{in^3}{min}$

Self-Assessment 36-25.3

1. If $z = x^3 - y$, $x = te^t$, $y = \sin t$, find $\frac{dz}{dt}$.
2. If $z = \frac{x^2}{y}$, $x = ue^v$, $y = u + e^v$, find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.
3. If $z = x^2y + x$ and $y = xe^x$, find $\frac{dz}{dx}$.
4. The power P in watts expended by a resistor of R ohms with a current of I amps is given by $P = I^2R$. At a certain instant in time, I = 40 amps and is increasing at the rate of 0.5 amp/sec, while R = 30 ohms and is decreasing at the rate of 0.4 ohm/sec. Find the rate of change of P with respect to time at that instant.

36-25.4 You will be able to use partial derivatives to determine derivatives and partial derivatives of implicit functions.

Premeasure

Find $\frac{dy}{dx}$ if

1. $x^2 + y^2 = 4$
2. $y^2 \sin x = 1 - \cos y$

Study Pages

Riddle, pages 680-682, Section 21.4

Learning Supplement

Note that we are not studying any material beyond page 682 in this section. Recall that from Chapter 4, Section 6, an equation in x and y could be interpreted as defining y as a function of x, and then it was possible to find $\frac{dy}{dx}$ by differentiati both sides of the equation with respect to x (see the Premeasure). Theorem 21.4 is just another way of finding $\frac{dy}{dx}$, this time using partial derivatives. Note that while we still consider y to be a function of x, for Theorem 21.6, we set $F(x,y) = 0$, where F is a function of the two independent variables x and y, and then determine $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ to find $\frac{dy}{dx}$.

Example: Find $\frac{dy}{dx}$ if $y^2 \sin x = 1 - \cos y$ (Premeasure 2)

Solution: Set $F(x,y) = y^2 \sin x + \cos y - 1 = 0$.

Then, $\frac{\partial F}{\partial x} = y^2 \cos x$, $\frac{\partial F}{\partial y} = 2y \sin x - \sin y$, and

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{y^2 \cos x}{2y \sin x - \sin y}.$$

Theorem 21.7 deals with the more complicated case where $F(x,y,z) = 0$ and z is considered to be a function of x and y, i.e., $z = f(x,y)$. So, we want to find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$. Theorem 21.7 shows how these can be found in terms of $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$.

(Note: F is a function of 3 variables.) The equation,

$$\frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

on page 681 is obtained by using the chain rule (Theorem 21.5 expressed for a function of three variables). Then since x,y are considered to be independent variables, $\frac{\partial y}{\partial x} = 0$. Also, $\frac{\partial x}{\partial x} = 1$, and it then follows that,

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

Example: If $x^2 - xyz + z^2 = 2$, find $\frac{\partial z}{\partial x}$, $\frac{\partial y}{\partial x}$

Solution: First, $F(x,y,z) = x^2 - xyz + z^2 - 2 = 0$

Considering z as a function of x and y,

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{2x - yz}{-xy + 2z} = \frac{yz - 2x}{2z - xy}$$

Considering y as a function of x and z,

$$\frac{\partial y}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{2x-yz}{-xz} = \frac{2x-yz}{xz}$$

Exercises

Riddle, page 687, Section 21.4, problems 1, 3, 5, 7, 9, 11

Self-Assessment 36-25.4

1. If $(x^2 + y)^2 + y = 2$, find $\frac{dy}{dx}$

2. If $\sin xy = \cos x + \cos y$, find $\frac{dy}{dx}$

3. If $(x+y)^2 = (y-z)^3$, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, and $\frac{\partial y}{\partial z}$

Unit 36-25.5

36-25.5 You will be able to determine the total differential of a function of two variables and to determine when a differential expression is an exact differential.

Premeasure

1. Find dy if $y = x^3 - x^2$.
2. Find dy if $x^2 + y^2 = 9$

Study Pages

Riddle, pages 688-690, Section 21.5 (through Example 3)

Learning Supplement

Recall from Section 6.3 the differential for a function $y = f(x)$ of one variable is defined as $dy = f'(x)dx$, and that dy is used to approximate $\Delta y = f(x+\Delta x) - f(x)$ when Δx, Δy are small and $dx = \Delta x$. It was also pointed out that dy is a function of the two independent variables x and dx, and dx could be any real number, but applications require that Δx and dx be small (see Diagram 6.15, page 147).

For a function $z = f(x,y)$ of two variables, the total differential is defined formally as:

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

It should always be remembered that dz is a function of the four independent variables x, y, dx, dy, and that nothing is said about dx and dy being small. However, in the next section, it will be shown how dz can be used to approximate $\Delta z = f(x + \Delta x, y + \Delta y) - f(x,y)$ when Δx, Δy, dx, dy are small.

The development of the concept of an exact differential in the text is quite good. Just be sure that you keep straight which variable you are using for differentiation or integration. Perhaps one more example would help.

Example: Show that $(2xy^3 + e^x)dx + (3x^2y^2 - 2y)dy$ is exact and find the function $F(x,y)$ for which it is the total differential.

Solution: Setting $P = 2xy^3 + e^x$, $Q = 3x^2y^2 - 2y$, $\frac{\partial P}{\partial y} = 6xy^2$, $\frac{\partial Q}{\partial x} = 6xy^2$ and the differential is exact. To find F, we must first integrate P with respect to x, regarding y as a constant. Thus,

$$F(x,y) = \int (2xy^3 + e^x)dx = x^2y^3 + e^x + c(y)$$

Note that the constant of integration is a function of y. To find $c(y)$, use $\frac{\partial F}{\partial y} = Q$.

$$\frac{\partial F}{\partial y} = 3x^2y^2 + c'(y) = Q = 3x^2y^2 - 2y$$

which implies $c'(y) = -2y$ or $c(y) = \int -2y dy = -y^2 + k$. Thus, $F(x,y) = x^2y^3 + e^x - y^2 + k$. You should check that $\frac{\partial F}{\partial x} = P$ and $\frac{\partial F}{\partial y} = Q$.

This is not the only way to find F. For example, you could first use $F(x,y) = \int Q dy$ and use $\frac{\partial F}{\partial x} = P$ to determine the constant of integration. (Try it.) Do not worry about Example 4, page 690.

Exercises

Riddle, page 691, Section, 21.5, problems 1, 3, 5, 7, 11, 17, 19

Self-Assessment 36-25.5

1. Find the total differential df for

   a. $f(x,y) = \frac{x}{y} + \ell ny$

   b. $f(x,y) = x\sin(x+y)$

2. Test for exactness. If the differential expression is exact, find the function $F(x,y)$ for which it is exact.

   a. $e^y dx + (xe^y - 2y)dy$

   b. $(2x^3y+3y^2)dx + (3x^2y^2 + 6x)dx$

Unit 36-25.6

36-25.6	You will be able to use the total differential in approximation problems.

Study Pages

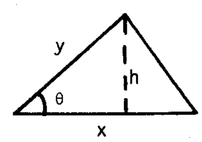
Riddle, pages 692-693, Section 21.6

Learning Supplement

Again, recall that for $y = f(x)$, $dy = f'(x)dx$ is used to approximate $\Delta y = f(x + \Delta x) - f(x)$ when Δx is small. Compare this to the situation for a function $z = f(x,y)$. In this case, $dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$, and dz is used to approximate $\Delta z = f(x + \Delta x, y + \Delta y) - f(x,y)$ when Δx, Δy are small. In applications, we set $dx = \Delta x$, $dy = \Delta y$. Study the examples in the text carefully. In Examples 1 and 2 note the use of the triangle inequality for absolute value, $|a+b| \leq |a| + |b|$ In Example 1, $|dA| = |ydx + xdy| \leq |ydx| + |xdy| = |y||dx| + |x||dy|$.

Example: Two sides of a triangle were measured as 150 ft and 200 ft, and the included angle as $60^0 00'$. If the possible errors are 0.2 ft for the sides and 10' for the angle, find the greatest possible error in the computed area.

Solution: For a triangle $A = \frac{1}{2}bh = \frac{1}{2}x(y\sin\theta) = \frac{1}{2}xy\sin\theta$.



Hence, A is a function of three variables (x,y,θ), but we can compute dA as in the case for two variables and use dA to approximate $\Delta A = A(x+\Delta x,\ y+\Delta y,\ \theta+\Delta\theta) - A(x,y,\theta)$.

Thus, $A = \frac{1}{2}xy\sin\theta = \frac{1}{2}(150)(200)(\frac{\sqrt{3}}{2}) \approx 12{,}990.4\ \text{ft}^2$,

$$\text{and } dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial \theta}d\theta$$

$$= (\frac{1}{2}y\sin\theta)dx + (\frac{1}{2}x\sin\theta)dy + (\frac{1}{2}xy\cos\theta)d\theta$$

If $x = 150$, $y = 200$, $\theta = 60^0$, $|dx| \leq .2$, $|dy| \leq .2$, and $|d\theta| \leq 10' = 0.0029$ radians (note that radian measure must be used), then

$$|dA| \leq \frac{1}{2}(200)(\frac{\sqrt{3}}{2})(.2) + \frac{1}{2}(150)(\frac{\sqrt{3}}{2})(.2) + \frac{1}{2}(150)(200)(\frac{1}{2})(.0029)$$

$$|dA| \leq 52.1\ \text{ft}^2.$$

By these measurements, then, $A = 12{,}990.4\ \text{ft}^2$ with a maximum error of $|dA| \leq 52.1\ \text{ft}^2$.

Exercises

Riddle, pages 693-694, Section 21.6, problems 1, 5, 7, 9, 13

Self-Assessment 36-25.6

1. Use differentials to approximate $\frac{12.2}{\sqrt{4.1}}$

2. A box with a square base was found to have a height of 10 cm, and the length of a side of its base was found to be 5 cm. If the maximum error in each measurement was .1 cm, find the volume of the box and a maximum error in this calculation of the volume.

3. Exercise 12, page 604, Riddle
   Use 30'' = .00015 radian, tan 23^0 = .4245, sec 23^0 = 1.0864, and a calculator.

Unit 36-25.7

36-25.7	You will be able to find the gradient of a function of two variables and the directional derivatives of a function of two variables.

Study Pages

Riddle, pages 694-698, Section 21.7

Learning Supplement

The partial derivates of $z = f(x,y)$ give us derivatives in two directions, the x-direction and the y-direction. Derivatives in the other directions are handled by the idea of directional derivatives. Notice how the definition of a directional derivative is similar to the definition of a derivative of a function of one variable. Here we are specifying that we must approach (x,y) in the direction determined by the vector $\mathbf{v}$, and hence we are working in one dimension. So, as $h \to 0$, $h\cos\theta \to 0$, $h\sin\theta \to 0$ which implies $(x + h\cos\theta,\ y + h\sin\theta) \to (x,y)$ along the line determined by $\mathbf{v}$. Thus,

$$D_{\mathbf{v}}z = \lim_{h \to 0} \frac{f(x + h\cos\theta,\ y + h\sin\theta) - f(x,y)}{h}$$

is similar to $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ because for $f'(x)$, $(x+h) \to x$ as $h \to 0$. Geometrically, $f'(x)$ is the slope of the curve $y = f(x)$ at $(x,f(x))$. Note how this is also similar to the geometric description of the directional derivative given in Riddle on the bottom of page 695.

Make sure you understand the relationship between directional derivatives and partial derivatives given on page 696. Theorem 21.10 also gives a relatively easy way to compute directional derivatives using partial derivatives. Using the gradient of f, $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ (page 698), Theorem 21.10 translates to $D_{\mathbf{v}}z = \nabla f \cdot \mathbf{v}$ if $\mathbf{v}$ is a unit vector. If $\mathbf{u}$ is not a unit vector, then $\frac{\mathbf{u}}{|\mathbf{u}|}$ is a unit vector and $D_{\mathbf{u}}z = \nabla f \cdot \frac{\mathbf{u}}{|\mathbf{u}|}$.

Example: If $z = x^3 + xy + y^3$ and $\mathbf{v} = \sqrt{3}\,\mathbf{i} - \mathbf{j}$, find $D_{\mathbf{v}}z$.

Solution: $|\mathbf{v}| = \sqrt{3+1} = 2$. So, $\mathbf{v}$ is not a unit vector.

$$\text{Hence, } D_{\mathbf{v}}z = \nabla f \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)\left(\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right)$$

$$= \left[(3x^2+y)\mathbf{i} + (x+3y^2)\mathbf{j}\right] \cdot \left[\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right]$$

$$= (3x^2+y)\frac{\sqrt{3}}{2} + (x+3y^2)\left(-\frac{1}{2}\right) = \frac{3x^2\sqrt{3} - 3y^2 - x + y\sqrt{3}}{2}$$

Note that $D_{\mathbf{v}}z$ can now be evaluated at any point $(x,y,f(x,y))$. In the above example, at $(1,0,1)$, $D_{\mathbf{v}}z = \frac{3\sqrt{3} - 1}{2}$.

Exercises

Riddle, pages 700-701, Section 21.7, problems 1, 3, 5, 11, 13, 15, 16

Study Pages

Riddle, pages 699-700, Section 21.7

Learning Supplement

Now that you are familiar with directional derivatives, it might be natural to ask, "if $z = f(x,y)$ and the point (x,y) is fixed, you can calculate any number of directional derivatives $D_{\mathbf{v}}f$ as the unit vector $\mathbf{v}$ varies. In what direction $\mathbf{v}$ is the value of $D_{\mathbf{v}}f$ a maximum?" The answer to this question is supplied by Theorem 21.12.

In the discussion following Theorem 21.12, note that the normal derivative of $z = f(x,y)$ is $\frac{df}{dn} = |\nabla f| = \sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}$, and this is the maximum value for $D_{\mathbf{v}}f$ and the direction for this maximum value of $D_{\mathbf{v}}f$ is $\mathbf{v} = \frac{\nabla f}{|\nabla f|}$.

Example: Find the maximum value of the directional derivative of $f(x,y) = x^2 - 6y^2$ at $(7,2,25)$ and determine its direction.

Solution: $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = 2x\mathbf{i} - 12y\mathbf{j}$. Thus,

$\frac{df}{dn} = |\nabla f| = \sqrt{4x^2 + 144y^2}$. So, at $(7,2,25)$ the maximum directional derivative is,

$$|\nabla f| = \sqrt{4(7)^2 + 144(2)^2} = \sqrt{772} = 2\sqrt{193} \approx 27.8$$

and the direction is given by the unit vector

$$\mathbf{v} = \frac{\nabla f}{|\nabla f|} = \frac{2x\mathbf{i} - 12y\mathbf{j}}{2\sqrt{193}} = \frac{14\mathbf{i} - 24\mathbf{j}}{2\sqrt{193}} = \frac{7\mathbf{i} - 12\mathbf{j}}{\sqrt{193}}$$

$$= (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$$

So, $\cos\theta = \frac{7}{\sqrt{193}}$, $\sin\theta = \frac{-12}{\sqrt{193}}$ and the direction is $\theta \approx 300^{\circ}15'$. (Note: It is really not necessary to determine θ since the vector $\mathbf{v}$ determines the direction.)

Exercises

Riddle, page 701, Section 21.7, problems 21, 23, 27, 31

Self-Assessment 36-25.7

1. Find the directional derivative of $z = e^{xy}$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

2. Find the directional derivative of $z = y\sin x$ in the direction of $\mathbf{v} = \mathbf{i} - \sqrt{3}\,\mathbf{j}$ at (0,2,0).

3. For $z = x^3y$, find the maximum directional derivate (normal derivative) and its direction (in terms of a unit vector $\mathbf{v}$) at the point (1,4,4).

Unit 36-25.8

36-25.8	You will be able to find the equations of the tangent plane and the normal line of a surface at a given point.

Learning Supplement

In studying functions of two variables it is instructive to consider what happens for a function of one variable and then see if and how this concept can be interpreted in the two variable case. For example, it has already been shown that a partial derivative of $z = f(x,y)$ (and a directional derivative) represents a slope of a curve formed by the intersection of the surface of $z = f(x,y)$ and a vertical plane. (For a partial derivative, this plane is parallel to either the x or y axis.) This is similar to the fact that a derivative of a function of one variable is the slope of a curve at a given point. Since the derivative of $y = f(x)$ can be used to determine the equation of a line tangent to $y = f(x)$, it seems reasonable to expect that the partial derivatives of $z = f(x,y)$ could be used to find the equation of a plane tangent to $z = f(x,y)$.

In order to understand the development of the equations for the tangent plane and the normal line, it is essential that you understand the concepts studied in Mathematics Module 57-36-20 (Riddle, Chapter 20), and, in particular, Theorems 20.17 and 20.24 in Riddle. Recall that in order to determine the equation of a plane, we need to know a point in the plane and a line l perpendicular to the plane. Also, if $\mathbf{u}, \mathbf{v}$ are two vectors in a plane, then the vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Study Pages

Riddle, pages 701-705, Section 21.8

Learning Supplement

The only results you need concern yourself with in this section are Theorems 21.13 and 21.14 which give a simple method to determine tangent planes of a surface. In the proof of Theorem 21.13, the vectors $\mathbf{u}, \mathbf{v}$ on page 702 lie in the tangent plane. The vector $\mathbf{v}$ lies in the plane $y = y_0$ and so has $\mathbf{j}$-component = 0 (since any two points in this plane have y-coordinate = y_0). If the $\mathbf{i}$-component is then set equal to one, then the k-component is $\frac{\partial z}{\partial x}$ since $\frac{\partial z}{\partial x}$ = slope of the tangent line (see Figure 21.6, page 703). Thus, $\mathbf{v} = \mathbf{i} + 0 \cdot \mathbf{j} + \frac{\partial z}{\partial x}\mathbf{k}$. Similarly, $\mathbf{u} = 0 \cdot \mathbf{i} + \mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k}$, since $\mathbf{u}$ is in the plane $x = x_0$. Next, $\mathbf{u} \times \mathbf{v}$ gives a vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ and hence perpendicular to the tangent plane. Then, Theorem 20.24 is applied to give the equation of the tangent plane. Theorem 21.14 is a very simple consequence of Theorem 21.13 and the methods of implicit differentiation (Theorem 21.7).

Exercises

Riddle, page 707, Section 21.8, problems 1, 3, 5, 9, 13, 15

Self-Assessment 36-25.8

Find equations for the tangent plane and normal line to the given surface at the indicated point.

1. $z = x^2 - xy$ at $(2,-1,6)$
2. $z = x^2 \cos y$ at $(2,0,4)$
3. $xyz + x + y + z = -3$ at $(1,-2,2)$

Unit 36-25.9

> 36-25.9 You will be able to determine the relative maxima, relative minima, and saddle points of a function or two variables.

Study Pages

Riddle, pages 708-710, Section 21.9

Learning Supplement

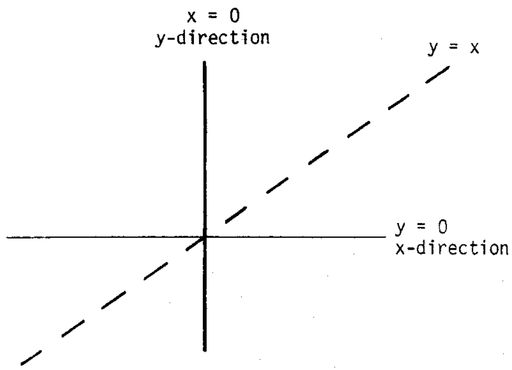
In the first paragraph of Section 21.9, the text states that in order to have a relative maximum or minimum, there must be a horizontal tangent plane. Thus, if $(a,b,f(a,b))$ is a relative max or min, the tangent plane has equation $z = f(a,b)$. Hence, by Theorem 21.13, $A = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = 0$ and $B = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = 0$. A critical point is defined to be a point where both partial derivatives equal zero, and in order to determine the critical points, you solve the simultaneous system of two equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. This corresponds to locating critical points by solving $f'(x) = 0$ in the one variable case.

In the one variable case it was sometimes possible to determine if a critical point was a rel. max or min by using the second derivative. The reason why the use of the second partial derivatives to determine if a critical point is a rel. max or min or neither is more complicated, is that there are many directions from which to approach that point, and $\frac{\partial^2 f}{\partial x^2}$. $\frac{\partial^2 f}{\partial y^2}$ deal with only two directions, parallel to the x-axis and parallel to the y-axis. They do not say anything about what happens in the other directions. A couple of examples should illustrate this.

Example 1: $z = f(x,y) = y^2 - x^2$. $\frac{\partial z}{\partial x} = -2x$, $\frac{\partial z}{\partial y} = 2y$

The only critical point is (0,0,0). In the x-direction, $\frac{\partial^2 z}{\partial x^2} = -2$, and (0,0,0) is a rel. max in the x-direction (second derivative < 0). (This means that (0,0,0) is a rel. max for the curve formed by the intersection of $z = f(x,y)$ and $y = b$, in this case, $z = y^2 - x^2$ and $y = 0$, which yields $z = -x^2$.)

In the y-direction, $\frac{\partial^2 z}{\partial y^2} = 2$ and (0,0,0) is a rel. min in the y-direction. Hence, by investigating only these two directions, it is seen that (0,0,0) is neither a rel. max nor a rel. min. In this case, (0,0,0) is called a saddle point. *Note:* By restricting ourselves to a line in the plane (in the above example, x=0 or y=0) we have changed the problem to a function of one variable, and we can use everything we know about a function of one variable to study the curve in the vertical plane determined by that line. In the next example, the line y=x will also be used.



Example 2: If $z = f(x,y) = 2x^2 - 5xy + y^2$, then $\frac{\partial z}{\partial x} = 4x - 5y$,

$\frac{\partial z}{\partial y} = -5x + 2y$ and (0,0,0) is the only critical point.

In the x-direction ($y = 0$), $\frac{\partial^2 z}{\partial x^2} = 4$, ($z = f(x,0) = 2x^2$),

and (0,0,0) is a rel. min. In the y-direction ($x = 0$),

$\frac{\partial^2 z}{\partial y^2} = 2$ and (0,0,0) is a rel. min. However, along the

line $y = x$, $z = g(x) = f(x,x) = -2x^2$, and $g''(x) = -4$

implies (0,0,0) is a rel. max in this direction.

Thus, (0,0,0) is a saddle point. (It cannot be a rel.

min, since $f(x,x) < 0 = f(0,0)$, for all $x \neq 0$.)

These two examples were for illustration purposes only. You will not usually have to use such an analysis. It is hoped that they illustrate the need to use something more complicated than just $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ which deal only with the x and y directions. You should now go back and do both examples using Theorem 21.16. Also, study the four examples in the text. However, you may have to do something similar to what was done in Example 2 in the rare case that $D = 0$.

Example 3: Test $z = f(x,y) = \frac{4}{x} + \frac{2}{y} + xy$ for rel. max and min.

Solution:
$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \frac{-4}{x^2} + y \\ \frac{\partial f}{\partial y} &= \frac{-2}{y^2} + y \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \frac{-4}{x^2} + y &= 0 \\ \frac{-2}{y^2} + x &= 0 \end{aligned}$$

Using the first equation, $y = \frac{4}{x^2}$, and substituting into the second, $x = \frac{2}{y^2} = \frac{x^4}{8}$. Thus,

$$x^4 - 8x = 0 \text{ or } x(x^3 - 8) = 0$$

$$x(x-2)(x^2 + 2x+4) = 0.$$

x cannot be zero and so x = 2 is the only valid solution and (2,1,6) is the only critical point.

$$\frac{\partial^2 f}{\partial x^2} = \frac{8}{x^3}, \quad \frac{\partial f}{\partial y} = \frac{4}{y^3}, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = 1$$

Thus, at (2,1,6) $D = (\frac{8}{8})(\frac{4}{1}) - 1^2 = 3 > 0$. Also, $\frac{\partial^2 f}{\partial x^2} = 1$, $\frac{\partial^2 f}{\partial y^2} = 4$ imply (2,1,6) is a rel. min.

Example 4: Test $z = f(x,y) = x^3 + x^2y + y^2 - 4y + 2$ for rel. max and rel. min.

Solution: $\frac{\partial f}{\partial x} = 3x^2 + 2xy$, $\frac{\partial f}{\partial y} = x^2 + 2y - 4$

To find the critical points, start with $3x^2 + 2xy = 0$ which implies $x(3x + 2y) = 0$, i.e., $x = 0$ or $2y = -3x$. Then,

(a) $x = 0$ and $x^2 + 2y - 4 = 0$ yields $y = 2$.
    (0,2,-2) is a critical point.

(b) $2y = -3x$ and $x^2 + 2y - 4 = 0$ yields $x^2 - 3x - 4 = 0$ or $(x-4)(x+1) = 0$. Thus, $x = -1$ or $x = 4$, and $(-1, \frac{3}{2}, \frac{-5}{4})$ and (4,-6,-10) are critical points.

$\frac{\partial^2 f}{\partial x^2} = 6x + 2y$, $\frac{\partial^2 f}{\partial y^2} = 2$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 2x$

and $D = (12x + 4y) - 4x^2$. Thus,

(1) At $(0,2,-2)$, $D = (4)(2) - 0 > 0$ and $(0,2,-2)$ is a rel. min.

(2) At $(4,-6,30)$, $D = -40$ and $(4,-6,-10)$ is a saddle point.

(3) At $(-1, \frac{3}{2}, \frac{-5}{4})$, $D = -10$ and $(-1, \frac{3}{2}, \frac{-5}{4})$ is a saddle point.

Exercises

Riddle, pages 710-711, Section 21.9, problems 1, 3, 5, 7, 8, 13, 15, 17, 21

Solution to Number 8:	$(3,0,27)$: $D = -144$	saddle point
	$(3,4,-5)$:	rel. min.
	$(-1,0,-13)$:	rel. max.
	$(-1,4,-45)$: $D = -144$	saddle point

Hint for Number 21: Use the distance formula and get $d = \sqrt{(x-1)^2 + (y-3)^2 + (z-4)^2}$. Substitute $z = 1 - 2x + y$ and minimize $D = d^2$.

Self-Assessment 36-25.9

Investigate for relative maxima and minima.

1. $z = 4x^2 - 8x - 4xy - \frac{y^2}{2} + 4y$
2. $z = e^{x^2+y^2}$ (Use the fact that $e^u \neq 0$ for all u.)
3. $z = x^2y + xy^2 - 3xy$

Answer Section

Premeasure 36-25.1

1. $\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$

2. $\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$

3. $\lim_{x \to 0} \frac{|x|}{x}$ does not exist

Self-Assessment 36-25.1

1. Along $x = 0$, $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y \to 0} \frac{-y^2}{y^2} = -1$

   Along $y = 0$, $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x \to 0} \frac{x^2}{x^2} = 1$

2. Along $y = 2x$, $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x \to 0} \frac{x^2 - 4x^2}{x^2 + 4x^2} = \lim_{x \to 0} \frac{-3x^2}{5x^2} = \frac{-3}{5}$

3. Along $x = 0$, $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{e^x \sin y}{x+y} = \lim_{y \to 0} \frac{\sin y}{y} = \lim_{y \to 0} \frac{\cos y}{1} = 1$

   Along $y = 0$, $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{e^x \sin y}{x+y} = \lim_{x \to 0} \frac{0}{x} = 0$

Self-Assessment 36-25.2

1. $\frac{\partial z}{\partial x} = y\cos y^2 \quad \frac{\partial z}{\partial y} = x\left[\cos y^2 - 2y^2 \sin y^2\right]$

   At (1,0,1), x=1, y=0 and $\frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial y} = 1\left[1 - 0\right] = 1$.

2. $\frac{\partial z}{\partial x} = e^{\frac{x}{y^2}} + \frac{x}{y^2}e^{\frac{x}{y^2}}$, $\frac{\partial z}{\partial y} = \frac{-2x^2}{y^3}e^{\frac{x}{y^2}}$

At (0,1,1), x = 0, y = 1 and $\frac{\partial z}{\partial x} = e^0 + 0 = 1$, $\frac{\partial z}{\partial y} = 0$

3. $\frac{\partial z}{\partial x} = \frac{\cos y}{y^2}$ $\quad \frac{\partial z}{\partial y} = x\left[\frac{-\sin y}{y^2} - \frac{2\cos y}{y^3}\right] = -x\left[\frac{y\sin y + 2\cos y}{y^3}\right]$

$\frac{\partial^2 z}{\partial x^2} = 0$

$\frac{\partial^2 z}{\partial y^2} = -x\left[\frac{y^3(y\cos y + \sin y - 2\sin y) - 3y^2(y\sin y + 2\cos y)}{y^6}\right]$

$= \frac{-x}{y^6}\left[y^4\cos y - 4y^3\sin y - 6y^2\cos y\right]$

$= \frac{-x}{y^4}\left[y^2\cos y - 4y\sin y - 6\cos y\right]$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{-\sin y}{y^2} - \frac{2\cos y}{y^3} = \frac{\partial^2 z}{\partial x \partial y}$

4. $\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}$, $\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$

$\frac{\partial^2 z}{\partial x^2} = \frac{2(x^2 + y^2) - 2x(2x)}{(x^2 + y^2)^2} = \frac{-2x^2 + 2y^2}{(x^2 + y^2)^2}$

$\frac{\partial^2 z}{\partial y^2} = \frac{2(x^2 + y^2) - 2y(2y)}{(x^2 + y^2)^2} = \frac{-2y^2 + 2x^2}{(x^2 + y^2)^2}$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{0 - 2x(2y)}{(x^2 + y^2)^2} = \frac{-4xy}{(x^2 + y^2)^2}$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{0 - 2y(2x)}{(x^2 + y^2)^2} = \frac{-4xy}{(x^2 + y^2)^2}$

Premeasure 36-25.3

1. $\frac{dy}{dx} = \left[\cos(4x^2 + 2x)\right] \cdot (8x + 2) = (8x + 2)\cos(4x^2 + 2x)$

2. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot (2x\cos x^2) = 2xe^{\sin x^2}\cos x^2$

Self-Assessment 36-25.3

1. $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = 3x^2(te^t + e^t) - 1 \cdot (\cos t) = 3x^2te^t + 3x^2e^t - \cos t$

2. $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{2x}{y} \cdot e^v - \frac{x^2}{y^2}(1) = \frac{2xye^v - x^2}{y^2}$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{2x}{y}(ue^v) - \frac{x^2}{y^2}(e^v) = \frac{2xyue^v - x^2e^v}{y^2}$

3. $\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} = (2xy+1) + x^2(xe^x + e^x) = (2x^2e^x+1) + x^3e^x + x^2e^x =$

$3x^2e^x + x^3e^x + 1$

4. $\frac{dP}{dt} = \frac{\partial P}{\partial I} \cdot \frac{dI}{dt} + \frac{\partial P}{\partial R} \cdot \frac{dR}{dt} = 2IR \cdot \frac{dI}{dt} + I^2 \cdot \frac{dR}{dt}$

$= 2(40)(30)(0.5) + (40)^2(-0.4)$

$= 1200 - 640 = 560$

Premeasure 36-25.4

1. $2x + 2y\frac{dy}{dx} = 0$ implies $\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$

2. $2y\frac{dy}{dx}\sin x + y^2\cos x = \sin y\frac{dy}{dx}$

$\frac{dy}{dx}(2y\sin x - \sin y) = -y^2\cos x$

$\frac{dy}{dx} = \frac{-y^2\cos x}{2y\sin x - \sin y}$

Self-Assessment 36-25.4

1. $F(x,y) = (x^2 + y)^2 + y - 2 = 0$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{4x(x^2+y)}{2(x^2+y)+1}$$

2. $F(x,y) = \sin xy - \cos x - \cos y$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{y\cos xy + \sin x}{x\cos xy + \sin y}$$

3. $F(x,y,z) = (x+y)^2 - (y-z)^3$

$$\frac{\partial y}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{2(x+y)}{3(y-z)^2}, \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{2(x+y) - 3(y-z)^2}{3(y-z)^2}$$

$$\frac{\partial y}{\partial z} = -\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial y}} = -\frac{3(y-z)^2}{2(x+y) - 3(y-z)^2}$$

Premeasure 36-25.5

1. $dy = y'dx = (3x^2 - 2x)dx$

2. $2xdx + 2ydy = 0$

$2ydy = -2xdx$

$dy = \frac{-x}{y}\,dx$

Self-Assessment 36-25.5

1. (a) $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{1}{y}dx + (\frac{-x}{y^2} + \frac{1}{y})dy = \frac{1}{y}dx + (\frac{y-x}{y^2})dx$

(b) $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \left[\sin(x+y) + x\cos(x+y)\right]dx + x\cos(x+y)dy$

2. (a) $\frac{\partial P}{\partial y} = e^y = \frac{\partial Q}{\partial x}$. Exact.

$F(x,y) = \int P dx = \int e^y dx = xe^y + c(y)$

$\frac{\partial F}{\partial y} = xe^y + c'(y) = Q = (xe^y - 2y)$. Thus,

$c'(y) = -2y$ or $c(y) = -y^2 + k$

$F(x,y) = xe^y - y^2 + k$

(b) $\frac{\partial P}{\partial y} = 2x^3 + 6y$ $\qquad \frac{\partial Q}{\partial x} = 6xy^2 + 6$. Not exact.

Self-Assessment 36-25.6

1. $z = f(x,y) = \frac{x}{\sqrt{y}}$

If $x = 12$, $y = 4$, then $z = \frac{12}{2} = 6$. The errors involved are $dx = 0.2$, $dy = 0.1$.

An estimate of the error of z is $dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dz = \frac{1}{\sqrt{y}} dx - \frac{x}{2y^{\frac{3}{2}}} dy$

$= \frac{1}{\sqrt{4}} (0.2) - \frac{12}{2.4^{\frac{3}{2}}} (0.1) = .1 - .075 = .025$

Thus, $\frac{12.2}{\sqrt{4.1}} \approx z + dz = 6.025$

2. $V = x^2y$

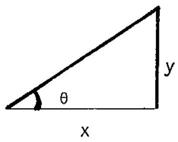
$x = 5$, $y = 10$, $|dx| \leq .1$, $|dy| \leq .1$

Thus, $V = 5^2 \cdot 10 = 250$ and $dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy$

$dV = 2xydx + x^2dy$ which implies $|dV| \leq |2xy| \cdot |dx| + |x^2| \; |dy| =$

$(100)(.1) + (25)(.1) = 12.5$

$V = 250 \text{ cm}^3$ with maximum error $|dV| \leq 12.5 \text{ cm}^3$

3. 

$\tan\theta = \frac{y}{x}$. x, θ are known. So $y = x\tan\theta$.

Thus, $dy = \frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial \theta}d\theta$ or

$dy = (\tan\theta)dx + (y\sec^2\theta)d\theta$, and

$|dy| \leq |\tan\theta|\,|dx| + |y\sec^2\theta|\,|d\theta|$ $\qquad x = 100,\ \theta = 23^0$

$|dx| \leq 0.1,\ |d\theta| \leq .00015$

$|dy| \leq (\tan 23^0)(0.1) + 100(\sec 23^0)^2(.00015)$

$= (.4245)(0.1) + 100(1.0864)^2(.00015) \approx .06$ ft.

$|dy| \leq 0.06$ ft. Note that $y = x\tan\theta = 100(.4245) = 42.45$ ft.

Self-Assessment 36-25.7

1. $\nabla z = \frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j} = (ye^{xy})\mathbf{i} + (xe^{xy})\mathbf{j}$

$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{\sqrt{9 + 16}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$. Hence,

$D_{\mathbf{v}}z = \nabla f \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3}{5}(ye^{xy}) - \frac{4}{5}(xe^{xy})$

2. $\nabla z = (y\cos x)\mathbf{i} + (\sin x)\mathbf{j}$, $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} - \sqrt{3}\,\mathbf{j}}{2}$. Thus,

$D_{\mathbf{v}}z = \frac{1}{2}(y\cos x) - \frac{\sqrt{3}}{2}(\sin x)$, and at $(0,2,0)$

$D_{\mathbf{v}}z = \frac{1}{2}(2\cos 0) - \frac{\sqrt{3}}{2}(\sin 0) = 1$

3. $\nabla z = (3x^2y)\mathbf{i} + x^3\mathbf{j}$. At $(1,4,4)$, $\nabla z = 12\mathbf{i} + \mathbf{j}$ and

$\frac{dz}{dn} = |\nabla z| = \sqrt{(12)^2 + (1)^2} = \sqrt{145}$. The direction is $\frac{\nabla z}{|\nabla z|} = \frac{12\mathbf{i} + \mathbf{j}}{\sqrt{145}}$

Self-Assessment 36-25.8

1. $\frac{\partial z}{\partial x} = 2x - y$, $\frac{\partial z}{\partial y} = -x$. At (2,-1,6), $\frac{\partial z}{\partial x} = 5$, $\frac{\partial z}{\partial y} = -2$

Equation of tangent plane is $5(x-2) - 2(y+1) - (z-6) = 0$ or $5x-2y-z-6 = 0$

Equation of normal line is $\frac{x-2}{5} = \frac{y+1}{-2} = \frac{z-6}{-1}$

2. $\frac{\partial z}{\partial x} = 2x\cos y$, $\frac{\partial z}{\partial y} = -x^2\sin y$. At (2,0,4), $\frac{\partial z}{\partial x} = 4$, $\frac{\partial z}{\partial y} = 0$

Equation of tangent plane is $4(x-2) - (z-4) = 0$ or $4x - z - 4 = 0$.

Equation of normal line is $\frac{x-2}{4} = \frac{z-4}{-1}$ and $y = 0$

3. $F(x,y,z) = xyz + x + y + z + 3 = 0$. At (1,-2,2),

$\frac{\partial F}{\partial x} = yz + 1 = -3$, $\frac{\partial F}{\partial y} = xz + 1 = 3$, $\frac{\partial F}{\partial z} = xy + 1 = -1$

Equation of tangent plane is $-3(x-1) + 3(y+2) - (z-2) = 0$ or $3x-3y+z-11 = 0$.

Equation of normal line is $\frac{x-1}{-3} = \frac{y+2}{3} = \frac{z-2}{-1}$

Self-Assessment 36-25.9

1. $\frac{\partial z}{\partial x} = 8x - 8 - 4y$, $\frac{\partial z}{\partial y} = -4x - y + 4$. Thus,

$\begin{cases} 2x-y-2 = 0 \\ 4x+y-4 = 0 \end{cases}$ and (1,0,-4) is the only critical point.

$\frac{\partial^2 z}{\partial x^2} = 8$, $\frac{\partial^2 z}{\partial y^2} = -1$, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = -4$. At (1,0,-4), $D = (8)(-1)-(-4)^2 < 0$

and (1,0,-4) is a saddle point.

2. $\frac{\partial z}{\partial x} = 2xe^{x^2+y^2}$, $\frac{\partial z}{\partial y} = 2ye^{x^2+y^2}$. So, $2xe^{x^2+y^2} = 0$ implies $x = 0$ since $e^{x^2+y^2} \neq 0$.

Similarly, $y = 0$ and (0,0,1) is the only critical point.

$\frac{\partial^2 z}{\partial x^2} = (4x^2+2)e^{x^2+y^2}$, $\frac{\partial^2 z}{\partial y^2} = (4y^2+2)e^{x^2+y^2}$, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = 4xye^{x^2+y^2}$

Hence, at (0,0,1), $D = (2)(2) - 0^2 = 4$ and (0,0,1) is a relative minimum.

3. $\frac{\partial z}{\partial x} = 2xy + y^2 - 3y$, $\frac{\partial z}{\partial y} = x^2 + 2xy - 3x$

$2xy + y^2 - 3y = 0$ gives $y(2x+y-3) = 0$, i.e., $y = 0$ or $2x + y = 3$

$x^2 + 2xy - 3x = 0$ gives $x(x+2y-3) = 0$, i.e., $x = 0$ or $x + 2y = 3$

There are four cases:

(a) $y = 0$, $x + 2y = 3$ or $(3,0,0)$ is a critical point

(b) $y = 0$, $x = 0$ or $(0,0,0)$ is a critical point

(c) $2x+y = 3$, $x = 0$ or $(0,3,0)$ is a critical point

(d) $\begin{array}{l} x + 2y = 3 \\ 2x + y = 3 \end{array}$ or $(1,1,-1)$ is a critical point

$\frac{\partial^2 z}{\partial x^2} = 2y$, $\frac{\partial^2 z}{\partial y^2} = 2x$, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = 2x + 2y - 3$

Thus, at

(a) $(3,0,0)$, $D = -9$, saddle point

(b) $(0,0,0)$, $D = -9$, saddle point

(c) $(0,3,0)$ $D = -9$, saddle point

(d) $(1,1,-1)$, $D = 3$, $\frac{\partial^2 z}{\partial x^2} = 2$, $\frac{\partial^2 z}{\partial y^2} = 2$, relative minimum

SELF-ASSESSMENT

1. Find all second partial derivatives of $z = e^{3x}\cos y$.

2. a) If $z = x^2 \sin y$, $x = ve^u$ and $y = u^2 + v^2$, find $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$

   b) If $e^{xy} = \cos(x^2 z)$, find $\frac{\partial z}{\partial x}$

3. Problem #32, page 680 in Riddle

4. a) Approximate $\sqrt{50}\,\sqrt[3]{26}$ using differentials.

   b) If $(8xy+3)dx + 4(x^2+ y^2)$ is an exact differential, find the function $F(x,y)$ for which it is the differential.

5. Find the directional derivative of $z = x^2e^y + \cos y$ in the direction $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

6. For $z = \frac{x^2}{y}$, find the maximum directional derivative (normal derivative) at $(2,1,4)$ and its direction in terms of a unit vector $\mathbf{v}$.

7. Find the equations for the tangent plane and normal line to the surface of $z^2 = x^2 + y^2$ at $(3,4,5)$.

8. Investigate for relative maxima, relative minima, and saddle points.

   (a) $z = f(x,y) = \frac{4}{x} + \frac{2}{y} + xy$

   (b) $z = f(x,y) = x^4 - 8x^2 + y^2 - 4y$

<u>Answers to Self-Assessment</u>

1. $\frac{\partial z}{\partial x} = 3e^{3x}\cos y$, $\frac{\partial z}{\partial y} = -e^{3x}\sin y$

$\frac{\partial^2 z}{\partial x^2} = 9e^{3x}\cos y$, $\frac{\partial^2 z}{\partial y^2} = -e^{3x}\cos y$, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = -3e^{3x}\sin y$

2. a) $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = (2x\sin y)ve^u + (x^2\cos y)(2u)$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = (2x\sin y)e^u + (x^2\cos y)(2v)$

b) $F(x,y,z) = e^{xy} - \cos(x^2 z)$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = \frac{ye^{xy} + 2xz\sin(x^2 z)}{x^2\sin(x^2 z)}$$

3. $A = xy$ $\frac{dA}{dt} = \frac{\partial A}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial A}{\partial y} \cdot \frac{dy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt} = (4\cdot 2 + 5\cdot 3)\frac{in^2}{min} = 23\frac{in^2}{min}$

4. a) $z = f(x,y) = x^{\frac{1}{2}}y^{\frac{1}{3}}$. Let $x = 49$, $y = 27$, $\Delta x = dx = 1$, $\Delta y = dy = -1$

Then, $z = \sqrt{49}\,\sqrt[3]{27} = 21$ and $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (\frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{3}})dx + (\frac{1}{3}x^{\frac{1}{2}}y^{-\frac{2}{3}})dy$

or $dz = \frac{1}{2}(49)^{-\frac{1}{2}}\ 27)^{\frac{1}{3}}(1) + \frac{1}{3}(49)^{-\frac{1}{2}}(27)^{-\frac{2}{3}}(-1) = \frac{3}{14} - \frac{7}{27} = \frac{-17}{378}$

Thus, $\sqrt{50}\,\sqrt[3]{26} \approx z + dz = 21 - \frac{17}{378} = 20 + \frac{361}{378} \approx 20.955$

b) $\frac{\partial P}{\partial y} = 8x = \frac{\partial Q}{\partial x}$: exact. Thus, $F(x,y) = \int P dx = 4x^2y + 3x + c(y)$.

Also, $Q = \frac{\partial F}{\partial y} = 4x^2 + c'(y)$ implies $c'(y) = 4y^2$ or $c(y) = \frac{4y^3}{3} + k$.

Hence, $F(x,y) = 4x^2y + 3x + \frac{4y^3}{3} + k$

5. $\nabla z = \frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j} = (2xe^y)\mathbf{i} + (x^2e^y - \sin y)\mathbf{j}$

$D_{\mathbf{v}}z = \nabla z \cdot \frac{\mathbf{v}}{|\nabla \mathbf{v}|} = \left[(2xe^y)\mathbf{i} + (x^2e^y - \sin y)\mathbf{j}\right] \cdot \left[\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\right] = \frac{1}{\sqrt{2}}(2xe^y + x^2e^y - \sin y)$

6. $\nabla z = \frac{2x}{y}\mathbf{i} - \frac{x^2}{y^2}\mathbf{j}$. At (2,1,4), $\nabla z = 4\mathbf{i} - 4\mathbf{j}$ and $\left.\frac{df}{dn}\right|_{(2,1,4)} = |\nabla z| =$ $\sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$. The direction is $\frac{\nabla z}{|\nabla z|} = \frac{4\mathbf{i} - 4\mathbf{j}}{4\sqrt{2}} = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$

7. $F(x,y,z) = x^2 + y^2 - z^2 = 0$. $\frac{\partial F}{\partial x} = 2x$, $\frac{\partial F}{\partial y} = 2y = \frac{\partial F}{\partial z} = -2z$

Thus, equation of tangent plane at (3,4,5) is $6(x-3) + 8(y-4) - 10(z-5) = 0$ or $3x + 4y - 5z = 0$. Equation of normal line is $\frac{x-3}{6} = \frac{y-4}{8} = \frac{z-5}{-10}$

8. a) $\frac{\partial f}{\partial x} = \frac{-4}{x^2} + y$, $\frac{\partial f}{\partial y} = \frac{-2}{y^2} + x$. Hence, $\frac{-4}{x^2} + y = 0$, $y = \frac{4}{x^2}$ and $\frac{-2}{\left(\frac{4}{x^2}\right)^2} + x = 0$

yields $x^4 - 8x = 0$, $x(x^3-8) = 0$. $x = 0$ or $x = 2$ and the critical point is (2,1,6). $\frac{\partial^2 f}{\partial x^2} = \frac{8}{x^3}$, $\frac{\partial^2 f}{\partial y^2} = \frac{4}{y^3}$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1$. Thus, at (2,1,6) $D = (1)(4) - 1^2 > 0$ and (2,1,6) is a rel. min.

b) $\frac{\partial f}{\partial x} = 4x^3 - 16x$, $\frac{\partial f}{\partial y} = 2y - 4$

$4x^3 - 16x = 0$ yields $x = 0$, $x = 2$, or $x = -2$

$2y - 4 = 0$ yields $y = 2$. Critical points are (0,2,-4), (2,2,-20), and (-2,2,-20)

$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 16$, $\frac{\partial^2 f}{\partial y^2} = 2$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$

Thus, at (0,2,-4) $D = (-16)(2) - 0^2 < 0$, saddle point

at (2,2,-20), $D = (32)(2)^2 - 0 > 0$, rel. min.

at (-2,2,-20), $D = (32)(2) - 0^2 > 0$, rel. min.

NOTES

COLLEGE IV MODULE EVALUATION FORM

TO ALL STUDENTS OF THIS MODULE:

To improve the modules of College IV so that they better meet YOUR need, you are urged to complete this module evaluation form. Any student (whether registered or not) who has completed their work on this module is asked to answer the following items. All responses will remain anonymous; do not return this form to your professor. In advance, thanks.

RETURN TO: Green boxes located in Commons Room--Lake Michigan Hall--Room #154 or to Faculty Offices--Room #121, or to Laboratory--Rm. #136
MAIL TO: Module Evaluation Project, College IV, GVSC, Allendale, Mi. 49401

Enter the name of the module and its number: ______________________
name number

Please answer the following questions by circling the appropriate response, or filling the blank.

1. Which one of the following statements BEST characterizes your involvement with this module?
   1. I am registered for this module and am officially classified as a College IV student.
   2. I am registered for this module and am classified at Grand Valley as a CAS, TJC, WJC, or Seidman student.
   3. Although I am not registered for this module, I am using it to study for another course or module.
   4. Although I am not registered for this module, I obtained a copy of it to learn more about the subject matter.
2. How do you personally rate your chances for passing the mastery exam?
   1. Excellent
   2. Good
   3. Fair
   4. Poor
   5. I have no idea
   6. I have already taken the exam
   7. I am not taking the exam
3. For my preparation and ability the level of difficulty of this module was
   1. Very easy
   2. Somewhat easy
   3. About right
   4. Somewhat difficult
   5. Very difficult
4. The workload for this module in relation to other modules of equal credit was
   1. Much lighter
   2. Lighter
   3. About the same
   4. Heavier
   5. Much heavier
5. In total, how many hours did you spend working on this module?

   __________ hours

6. What portions of the material in this module were the most difficult for you to understand?

7. What did you like best about this module?

(OVER, PLEASE)

MODULE EVALUATION FORM (CONT.)

Below are a series of statements referring to specific characteristics of this module and your experiences with it. Indicate your response to each statement by circling the appropriate number at the right according to the following scale:

NA: 0 = Not Applicable or Don't Know. The statement does not apply to this module, or I am unable to give a knowledgeable response.
SA: 1 = Strongly Agree with the statement.
A: 2 = Agree. I agree more than I disagree with the statement.
D: 3 = Disagree. I disagree more than I agree with the statement.
SD: 4 = Strongly Disagree with the statement.

			AGREE		DISAGREE	
		NA	SA	A	D	SD
8.	The objectives for this module were very clear........	0	1	2	3	4
9.	I found considerable agreement between the stated objectives of the module and what I actually learned....	0	1	2	3	4
10.	I easily understood the basic concepts of this module.	0	1	2	3	4
11.	I found the assigned readings stimulating and thought provoking..	0	1	2	3	4
12.	I always understood what the professor expected of me as a student.....................................	0	1	2	3	4
13.	I found that the examples in the module helped me.....	0	1	2	3	4
14.	Whenever I sought help from the professor, she/he was readily available for consultation....................	0	1	2	3	4
15.	Whenever I sought help from the tutor, she/he was readily available for consultation....................	0	1	2	3	4
16.	I found that the media and audio-visual aids helped me understand the subject material of the module......	0	1	2	3	4
17.	The self-assessment test at the end of a unit(s) was a good measure of what I learned......................	0	1	2	3	4
18.	I was able to easily obtain all the materials necessary to complete the module......................	0	1	2	3	4

19. What suggestions do you have for improving this module? You may want to include both ideas and complaints about content, use of media, the sequence of learning activities, and the assigned workload.

20. Taking all things into account, what overall rating do you assign to this module (Superior, Good, Fair, or Poor)? Please explain.