Sabbatical Leave Proposal for Fall 2006
William Dickinson - Department of Mathematics
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1. **Descriptive title of your project**

   Equal Circle Packing Software

2. **Goals and objectives**

   The main goal of this sabbatical request is to create a dynamic, internet accessible and user friendly Java program and applet to explore, manipulate, and visualize equal (two dimensional) circle packings in various domains. I propose to do this in a setting where I will have regular meetings with an expert in this area (Dr. Robert Connelly of Cornell University) who is familiar with the myriad of approaches used in this branch of mathematics. This will result in the following outcomes:

   - This program will contain features that will support a new direction of inquiry into equal circle packing. Traditional equal circle packing research focuses on how to pack equal circles into solid boundary regions. The new direction I will pursue are the packings of equal circles into boundaryless or semi-boundaryless regions. (For details of the program and the new direction, see section 3.)
   - My understanding of and expertise in the well-developed methods used in traditional equal circle packings will deepen. This will allow me to apply them in this new area and become better able to guide undergraduates in their research in this new area.
   - This program will become a primary tool for undergraduates conducting research in this area. This continues and develops my prior success practice of involving undergraduates in accessible mathematical research where a computer program is the primary tool. (For more details on this practice see section 5.)
   - I will discover additional research questions in this new area of equal circle packing that are suitable for undergraduate research.

3. **Project Plan**

   The mathematical field of circle packing is concerned with optimizing the number of equal sized disks that can fit into a fixed region, where the disks are not allowed to overlap and must be entirely contained in the region. This kind of problem often arises in practical situations. For example, if you have a rectangular box and you want to fill it with as many cans of soda as possible (where the bottoms of the cans must touch the bottom of the box) then you are really trying to solve this kind of problem. The cross-sections of the cans are disks which you want to pack as efficiently as possible into the region, which is a cross-section of the box.[1] This kind of packing problem was considered

[1] In the dining commons I have seen students attempt to solve this kind of problem when they attempt to fit as many dessert plates on their tray as possible. Also, the process of putting as many equal sized glasses as possible into the top rack of a dishwasher is another example of this kind of problem

in the early 20th century ([15], [14], [4]) then brought into the mainstream with the work of mathematicians like Goldberg [7], Schaer [13] and Gardner [6] in the 1970's. Since then, it has been an active area of research in the discrete mathematical community. Mathematicians have generalized this to consider the packings of disks (and other regular objects) into various domains. For example, numerous research papers have appeared on the optimal packings of disks into circles [11], the packings of solid spheres into cubes [8], packings of circles on the surface of a sphere [5], packings of disks into an equilateral triangle [10] and many other variations on this theme.[2]

The new direction that I will pursue and develop is packing of equal disks into a different kind of region that is almost missing from the literature. [3] I want to consider regions that are boundaryless or semi-boundaryless. For example, instead of packing disks into a square with a solid boundary, I want to pack disks into a region, which is a square, except the left and right sides of the square are identified with each other. This means that a disk can move off of the left side and come in from the right.[4]

This field of study is a very visual type of mathematics and exploring different arrangements is a necessary aid in forming conjectures and proving theorems. Early tools for exploring these optimal arrangements were actually coins ([6]). Currently, computers are used to implement different kinds of algorithms. These algorithms fall into two broad categories: maximization of functions and geometric/physical. Most of the computer programs implementing these algorithms are not available to the public. The two that are available are not usable for my research. One is written for a platform (Unix/Linux) that students and faculty at Grand Valley do not have easy access to and doesn't focus on the packing of equal circles. The other is written in a way that is not useful for the boundaryless regions in which I want to examine packings.

The programs that I will create will have several unique features that will allow me to explore these new types of boundaryless regions. They will include:

- the ability to adjust the ratio of the length to width of the parallelogram region
- the ability to identify the left and right sides of the parallelogram region, or top and bottom of

[2]For beautiful illustrations of many of these packings, see the web page,
`http://www.stetson.edu/~efriedma/packing.html.`

[3]There is one article partially using this type of packing ([9]). However the authors present only their results on the torus with a large numbers of disks (2000) and appear to only have been interested in the large scale patterns and not proving the optimal density of arrangements for small numbers of disks. The optimality of the arrangements for small numbers of disks are the type of problem I'm interested in and will design this program for.

[4]This type of identification occurred in many early computer games, like Space Invaders. In that game, the defending ship could move continually to the right. When it got to the right most part of the screen it would appear on the left side of screen and continue moving right. Also the attacking ships in Galaga could move continually down by falling off the bottom part of the screen and appearing at the top of the screen.

the parallelogram region.

- the ability to reverse the orientation of the identifications of the sides
- the ability to adjust the angle between the sides.

In addition to these unique features the program will have many more standard features including the abilities to manipulate the circles, to display associated graphs and diagrams, and to access an algorithm that will dynamically increase the radii of the circles in a given domain.

4. **Timetable**

Pre-sabbatical: Consult with Dr. David Austin (an expert in mathematical Java programing) about the specifics of the implementation of the program.

September 2006: Move to Ithaca, New York and begin meeting with Dr. Connelly. Initially, discuss current ideas in packing theory and how they apply to packings in the torus. Begin writing the skeleton of the program.

October 2006: Have the program in a state in which the width/length ratio of the region, the angle between the sides, and identifications between sides can all be adjusted. Also, adding, removing and moving circles in the region will be possible. In consultation with Dr. Connelly, decide on an algorithm or series of adjustable algorithms to implement in order to grow the circles in the domain. In addition, during this month and the next one, I will be engaged in mathematical research surrounding circle packing as I explore possible algorithms and search out the problems accessible to undergraduates.

November 2006: Implement the algorithm or algorithms in the program. Continue meeting with Dr. Connelly and formally collect undergraduate research questions and possibly methods with which to approach these questions.

December 2006: Put finishing touches on the program. Return to Grand Valley.

Post Sabbatical: Either in the university's S^3 program or in the NSF's REU mathematics program hosted by the mathematics department, I will mentor an undergraduate research project in this new area using this program as a main tool. I will also present my work in the CLAS seminar, mathematics department seminar, and possibly other regional and national conferences (like the Michigan MAA sectional meeting or the annual Joint Meetings of the MAA and AMS). Also the program will be published to the web.

5. **Evidence of preparation.**

There are three areas in which I have experience that will enable me to successfully complete this project. There are:

- **Java programming experience:** In addition to my PhD in mathematics from the University of Pennsylvania, I simultaneously earned an MSE in Computer and Information Science. Therefore, I have had some formal training in computer science and specifically in the Java programming language.

  In practice, I was a coauthor of the Java program and applet *Spherical Easel*. This is a major Java application (freely available over the internet, see [1]) that Dr. David Austin and I created over the summer of 2002 to model spherical geometry. This program has been quite successful and is now in use as a teaching and research aid at the following institutions: Elmira College, York University (Canada), University of Illinois at Urbana-Champaign, Appalachian State University, Saint Louis University, Babbage Research Center (Buenos Aires, Argentina) and Framingham State College. In addition, Dr. David Austin will be availible to consult about any program specfic issues.

- **Mathematical knowledge of equal circle packings:** My interest in this area began as an undergraduate (in 1993-94) at Cornell University. I worked with Dr. Connelly in the area of equal circle packing on the torus for an academic year. This resulted in an unpublished honors thesis ([2]) in this area.[5] After finishing my undergraduate degree, I went to the University of Pennsylvania, where there was no faculty member who could serve as an advisor in the area of equal circle packing. Therefore, I got my degree in a different area of geometry, but I always kept my interest in equal circle packing and have attempted to keep tabs on the developments in the field. I have always been looking forward to the time when I could focus on this area professionally.

- **Mentoring of undergraduate research:** I have a successful track record of mentoring undergraduate research.
  - Student Ryan Koesterer and I obtained a division of mathematics and sciences SURP grant to study spherical triangle theorems. Part of our work from the summer (2002) was published in the peer reviewed journal for undergraduate research, The Pi Mu Epsilon Journal [3].
  - Student Kristina Lund and I obtained a university Student Summer Scholars (S^3) grant to study spherical geometry. Part of our summer (2003) work has been submitted to the peer reviewed journal, Mathematics Magazine. I'm very hopeful that this work will also be published. In addition, Kristina won an award for "Best Presentation" at a national summer meeting of the Mathematics Association of America (MAA) in Boulder, Colorado.

[5] An attempt to publish it (in the American Mathematical Monthly) resulted in a referee stating that it contained some nice observations but too little new mathematics to be published. I was pleased that it made it to the review process and now agree with the assessment of the referee.

- Along with three other faculty members in the mathematics department, I was a mentor in the National Science Foundation's Research Experiences for Undergraduates (REU) in mathematics. I mentored two students in the area of spherical and hyperbolic geometry. My student, Matt Katchke, won an award for his presentation "San Gaku Problems in Other Geometries" in Providence, Rhode Island, during the summer of 2004 at the national meetings of the MAA.

I have a good understanding of how to mentor undergraduate research. I know what kinds of questions are appropriate for undergraduates and the kind of support they need. I know that the area of equal circle packing is an area of mathematical research that is accessible to undergraduates. Hence, as my professional interests lay in both circle packing and in the mentoring of undergraduate research, equal circle packing is the perfect area to develop professionally. However, I know that I need a tool to make this area more accessible to undergraduates and the program I propose to write is just the right tool. Spherical Easel was an indispensable tool in the research that I conducted with undergraduates over the summers of 2003 and 2004. I know that the proposed program will served as a very useful research aid for the undergraduate research I plan to mentor in the future.

6. **Arrangements with people or other institutions**

   See the attached letters from

   - Dr. Ken Brown, chair of the mathematics department at Cornell University, stating that he will provide office space and access to Cornell's facilities.
   - Dr. Robert Connelly, Cornell University, stating that he will be at Cornell during the fall semester of 2006, and is willing to meet with me during this period.
   - Dr. David Austin, stating that he is willing to consult with me on this project.

7. **Benefit to other units**

   The mathematics department and the university are committed to undergraduate education and one component of that is engaging talented and interested students in research. This project will support this part of the university's mission and contribute to the community of scholars at the university.

   In addition to presenting my results in the required Grand Valley forums, I will present my work at regional and national conferences. I am a regular speaker at the Michigan section of the Mathematical Association of America and regularly attend the national meetings of the American Mathematical Society. I plan to disseminate my work at these conferences.

8. **Curriculum vitae**

   See attached.

References

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