Sabbatical Report
"Studies in Orthogonal Polynomials and Matrix Theory"
Academic Year 2004-05

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I stated five goals in my sabbatical proposal: (a) to deepen my understanding and expertise in my current research program, (b) to learn new mathematics in the areas of real analysis and linear algebra, (c) to develop, formulate, and solve new research questions, particularly for future joint work with undergraduates, (d) to gain additional perspective on my growth as a teacher by teaching a small number of courses at Hope College, and (e) to build stronger relationships between the mathematics departments at Hope and GVSU for future collaboration. I am pleased to report that I met all of these goals, and in addition engaged in several other significant scholarly endeavors during my sabbatical.

The early focus of my sabbatical was simply to read as much as possible. Over the past several years, I had collected a large number of papers that looked to be both interesting and relevant to my current research. In the course of my sabbatical, I read more than two dozen research articles carefully. Several of the papers were related to links between orthogonal polynomials and matrix theory; in one exciting article, I learned a construction of G. Schmeisser [1] that allows one to find a symmetric, tridiagonal companion matrix for a polynomial with all real zeros. This has the potential to allow me to use matrix analysis tools to analyze issues of interest in the polynomial functions that form the center of my research. Another paper I read was a lengthy 55 page survey titled "Hausdorff Geometry of Polynomials" [2], published in 2001, that contains an exciting new perspective on a long standing problem called the Sendov Conjecture. This open problem has spawned a large number of related problems, some of which I worked on during the year, and one of which was considered as part of my work with undergraduate researchers in Summer 2005. This paper and the corresponding area of mathematics will play a key role in my ongoing investigations in the coming years.

The reading and studying I accomplished led me to write a technical report on all that I had learned. By spring 2005, this document grew to nearly 60 pages. This manuscript provides background and perspective, including a number of key proofs, on the area of mathematics in which I work, and will be document that I will regularly use (in small pieces) to introduce undergraduate students to the area. In addition, throughout the document I regularly pose questions that form the basis for potential undergraduate research projects, as well as continued research for myself personally. The document concludes with six 1-2 page appendices that outline very specific possible projects for undergraduates to pursue, including suggested reading, open problems, and goals for each project. These questions and appendices will be a valuable resource to me as I lead future undergraduate research projects.

All of these activities together have most certainly deepened my expertise and understanding, broadened my knowledge through the learning of new mathematics, and led to the development of a significant number of potential future projects for work with students.

I taught two courses at Hope College during the course of the year: Multivariable Mathematics II in Fall 2004 (a mix of parts of GVSU's Maths 203, 300, and 304), and Calculus II (nearly identical to GVSU's Math 202) in Winter 2005. I had regular discussions with other Hope faculty about

teaching these courses. While the entire experience was at least satisfactory, teaching there made me miss the environment at GVSU and my colleagues here. Though most of the math professors at Hope are good teachers, they are almost exclusively lecturers. There is little time in their courses spent on learner-centered activities like group discovery activities in class or computer laboratory exercises; in addition, there seemed to be far less importance placed on mathematical communication and writing than we do here at GVSU. The focus seemed much more on "coverage" (i.e. lecture on absolutely as much information as possible) than on "understanding" (i.e. work to get students to truly understand as much of the subject as possible). It is certainly more work to teach in the "reform" style that is the norm in the GVSU math department, but having now also taught in an environment where students are used to a more "traditional" lecture-style presentation, I am left to conclude for many reasons that the progressive approach is genuinely superior for building student understanding, critical thinking, and communication skills.

The members of the Hope math department were pleasant and enjoyable colleagues for the year. Being there four days a week, attending (and organizing for one semester) their weekly colloquium, and participating most Fridays in a departmental lunch gave me the opportunity to foster several professional friendships. This summer I invited Professor Aaron Cinzori to speak to the GVSU REU group about his research with students at Hope College, and now being much more aware of the work being done by others, I look forward to continued interactions with them, at least partly through inviting them to speak at GVSU as opportunities arise.

In addition to the noted ways above that I specifically addressed the goals of my sabbatical as stated in the proposal, I was pleased to accomplish a variety of other important professional endeavors, including

- Finishing the work on a paper begun in August 2003 as a result of my work with students B. Vugteveen (GVSU) and J. Miller (Bucknell University) in the 2003 GVSU REU. This paper, "From Chebyshev to Bernstein: A Tour of Polynomials Large and Small" was submitted in August 2004 to the *College Mathematics Journal*, a widely read expository journal that is published by the Mathematical Association of America. The paper is still in review.
- Substantially revising and resubmitting two papers that had previously been rejected by other journals. Both these papers, "On the Ratio Vectors of Certain Families of Polynomials" and "A Note on PIPCIRs" are the result of joint work with former GVSU student Matthew Wells in a 2001 SURP grant and continued work the following year. The former paper was submitted to the *Missouri Journal of Mathematical Sciences* in December 2004, the latter to the *Pi Mu Epsilon Journal* in November 2004. Both papers are still in review.
- Writing a new paper titled "When Students Write the Text." This is a scholarship of teaching article that is an outgrowth of a project I conceived and implemented in our junior level Euclidean geometry course – we teach the class without a textbook and now have students write their own along the way through a structured and intense group project. The paper was submitted to *PRIMUS: Problems Resources and Issues in Mathematics Undergraduate Studies* in November 2004. It was accepted in January 2005 and appeared in print in June 2005.
- Developing several new talks, including a short tenure-reflection talk given in the dean's colloquium series at GVSU, a talk on the paper "When Students Write the Text" given at the fall MCTM (Michigan Council of Teachers of Mathematics) Conference and a slightly

different version at the Michigan MAA Section Meeting, and a talk on my research given to the GVSU REU in June 2005.

- Providing detailed editing and review of three research articles written by colleagues: papers by Will Dickinson and student Kris Lund (both of GVSU), Paul Fishback and student Nate Burch (both of GVSU), and Tom Pfaff (of Ithaca College).
- Completely editing and reviewing the new voting theory book of my colleague Jon Hodge (and his coauthor Rick Klima) prior to the text being submitted to the publisher for printing. All 200+ pages were carefully read and multiple editing and organizational suggestions were offered.
- Assisting colleague Ed Aboufadel in writing solutions for the textbook *Advanced Engineering Mathematics*, which he co-authored with M. Potter and J. Goldberg. I wrote solutions to *Maple* exercises for 3 chapters of the text.
- Co-authoring a textbook proposal that has been accepted by Oxford University Press, with M. Potter (MSU (retired)) and J. Goldberg (UM (retired)). (We still await the final details of a contract.) The book is a complete revision (with many parts started from scratch) of an earlier text they wrote on differential equations: *Differential Equations: A Systems Approach*. The new book, *Differential Equations with Linear Algebra*, is devised to (a) reach a broader audience, and (b) be usable as the text for a common "hybrid" class in both linear algebra and differential equations (like GVSU's Math 302), as well as for stand-alone differential equations courses. In May I began writing the first chapter, and have to date completed 6 sections in the first chapter for about 65 pages of text. This will be an ongoing 2+ year project to see to completion.

In addition to all these professional activities, I maintained electronic contact with my GVSU advisees, and also wrote letters of recommendation for 6 students applying to graduate school and 1 applying for an S^3 grant, for 2 students applying for scholarships, for 2 colleagues nominated for teaching awards, and for 3 contract renewals or promotions in math department personnel matters. I also organized the Hope College Mathematics Colloquium for one semester, and co-organized the 6th Annual Michigan Project NExT Symposium, a professional development opportunity for new faculty in mathematics.

In my post-sabbatical plans I wrote "Within the year or two following my sabbatical, I expect to engage in an undergraduate research project with one or more students on the subject of my study. I will also give a talk on my research in the mathematics department seminar and write the required report to fulfill Grand Valley's sabbatical requirements. Finally, I will attempt to publish at least one paper." This past summer, from June 13 – August 6, 2005, I was one of four faculty mentors in the NSF-funded GVSU Mathematics REU (Research Experience for Undergraduates). At this point in time, having just completed the REU, I am delighted to report that my work with students Justin From (Central College, IA) and Samuel Kolins (Bowdoin College) was tremendously successful, and was a direct outgrowth of my foregoing sabbatical.

One of the new problems that I studied during my sabbatical is a conjecture of R. Robinson on the maximum span of the derivatives of a polynomial with all real zeros. This conjecture, first published in 1967 [3] and then re-announced in 1988 [4] regards an accessible problem for which there is natural intuition, on which some significant progress has been made. This problem and the mathematics surrounding it is an area that I spent considerably time thinking about and preparing a plan for using it as a joint project with undergraduates. Near the end of my sabbatical I decided that this problem would be an excellent one to present to REU students, that it would be receptive to

applications of tools and results I had used in the past, and so I selected this problem as a central part of the summer 2005 project.

Through a sequence of carefully chosen journal articles (together with excerpts from the manuscript that I wrote during my sabbatical), Sam and Justin were led to consider this area of mathematics and several key related questions. In our collective efforts on this problem, we proved a collection of 6 theorems and lemmas; perhaps the main result is the Polynomial Root Squeezing Theorem which provides new insight into how the critical numbers of a polynomial are implicit functions of the polynomial's roots. One consequence of this work is that we were able to provide new and alternate proofs of results on the span of a polynomial and its derivatives. We are currently writing a journal article titled "The Polynomial Root Squeezing Theorem" that will be submitted this fall to the *American Mathematical Monthly*, a journal published by the Mathematical Association of America which is perhaps the most prestigious publication of general readership in mathematics.

I am delighted to have already had such a successful and tangible experience that was made possible by my year long sabbatical. Certainly I look forward to additional research in the area, as well as to more work with supervising students in the GVSU REU or in an S^3 project. To round out my post-sabbatical plans of mentoring students, writing a paper, and giving at least one talk, I will give a seminar talk to report on my sabbatical in the Mathematics Department Seminar in early September; further, I expect to speak in several other settings, including the Michigan Section of the MAA annual meeting in May 2006 on the research project I supervised in the 2005 REU.

Finally, I would like to express my thanks to my colleagues in the Department of Mathematics, to the College of Liberal Arts and Sciences, and to the University at large for the encouragement and support (including the financial support of half salary and benefits) that made a year of more focused scholarly investigation possible. In addition, I express my gratitude to Hope College and its Department of Mathematics for their financial support as well, in particular for the $26,000 salary (essentially the remaining half of my 2004-05 $52,500 salary) they paid me for teaching two courses and serving in their department.

References

[1] G. Schmeisser, A Real Symmetric Tridiagonal Matrix with a Given Characteristic Polynomial, *Lin. Alg. Appl.*, **193** (1993), 11-18.

[2] Bl. Sendov, Hausdorff Geometry of Polynomials, *East J. Approx.* **7** (No. 2) (2001), 123-178.

[3] R. Robinson, On the Spans of Derivatives of Polynomials, *Amer. Math. Monthly*, **71** (1964), 504--508.

[4] R. Robinson, Three Old Problems about Polynomials with Real Roots, *Amer. Math. Monthly*, **95** (1988), 329--330.