13

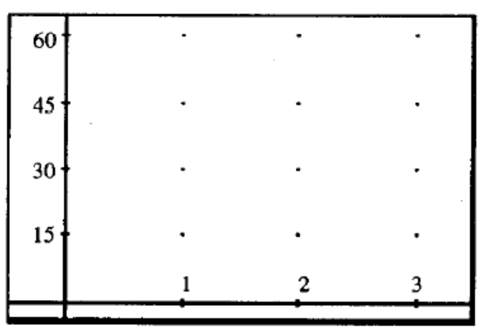
Determining Distance from Velocity

13.1 Introduction

The velocity function can obtained from the first derivative of a distance function. When the velocity function is provided, how might the distance function be determined? This exploration considers the relationship between a distance function and a velocity function when the velocity function is given.

13.2 Andy's Travel

1. Suppose that Andy travels at a constant rate of 60 mph for 3 hours.
   a. What is Andy's velocity at any moment during the time period from $t = 0$ to $t = 3$?
   b. Write an algebraic expression for the velocity function that models this situation.
   c. How far does Andy travel in these three hours?

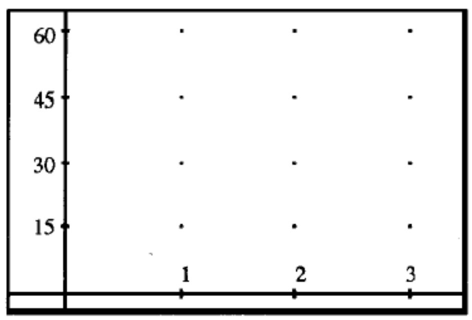


   d. Sketch the graph of the velocity function for t in [0, 3] on the axes provided above.

e. Sketch a vertical line from the t-axis to the graph of the velocity function through $t = 3$. What kind of geometric figure is sketched? How does the total distance Andy traveled relate to the graph of his velocity function? Explain.

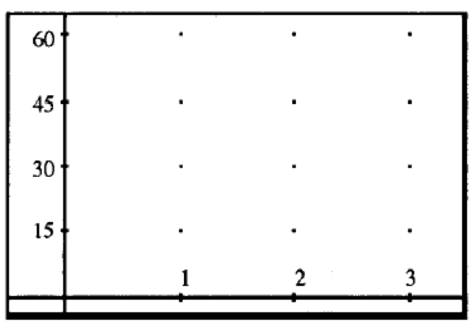
2. Suppose Andy traveled 60 mph for 2 hours, then traveled for 1 more hour at 45 mph.

   a. Sketch the graph of the velocity function for t in $[0, 3]$, labeling both the axes and the graph of the function.



   b. What is Andy's velocity at any moment during the first 2 hours? At any moment during the last hour?

   c. Write an expression that gives the velocity function that models this situation. (Use a piecewise-defined function.)



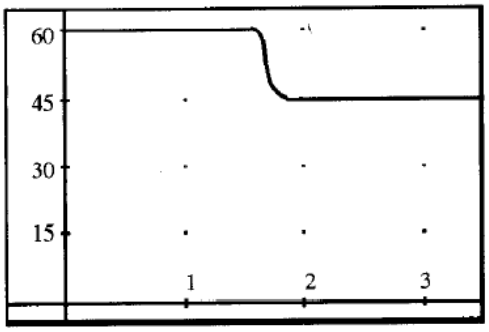
   d. How far does Andy travel in these 3 hours?

   e. Can the total distance Andy traveled be determined from the graph of his velocity function alone? Explain.

   f. Describe Andy's velocity at exactly 2 hours after his travel begins. Is this reasonable? Explain. If this situation is not reasonable, sketch a graph which models a more reasonable situation.[1]

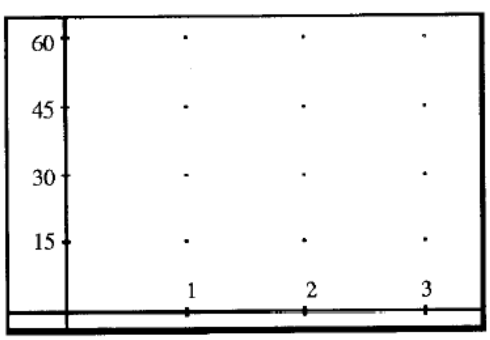
[1] It is important that you understand how your numerical and graphical results relate to each other. Discuss your findings with other members of your class. In particular, share your graphs drawn as a result of part (2f) and discuss the answer to part (2e) in light of your new graphs.

3. Following is a more realistic rendering of the situation described in part (2).



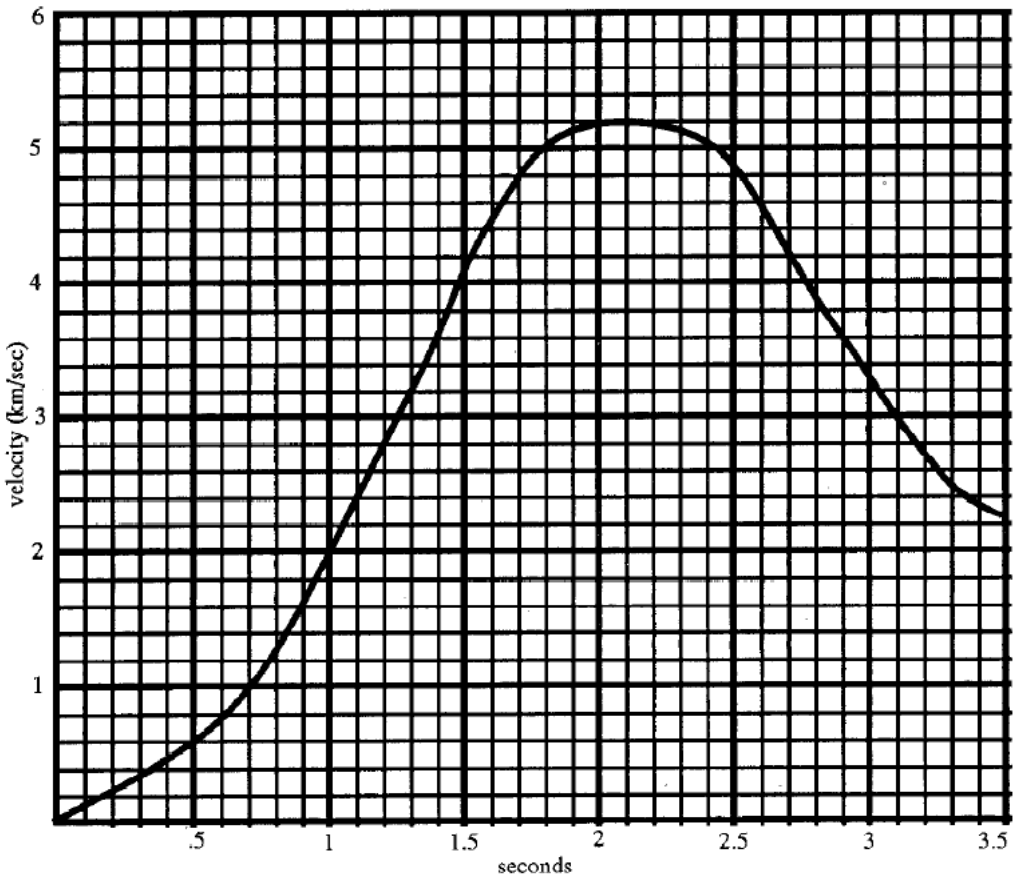
   a. Is it possible to determine the distance Andy traveled alone, from this graph? Explain.
   b. Suggest methods that could be used to determine the distance Andy traveled from the sketch of the graph.
   c. Use one of the methods suggested in (3b) to obtain a rough estimate of the distance Andy traveled.
   d. How might a more accurate estimate of the distance Andy traveled be obtained? Find this better estimate.

4. Suppose Andy's velocity is decreasing at a constant rate from 60 mph to 0 mph over a period of 3 minutes.
   a. Sketch the graph of the velocity function for t in $[0, 3]$. Label the axes and the graph of the function.



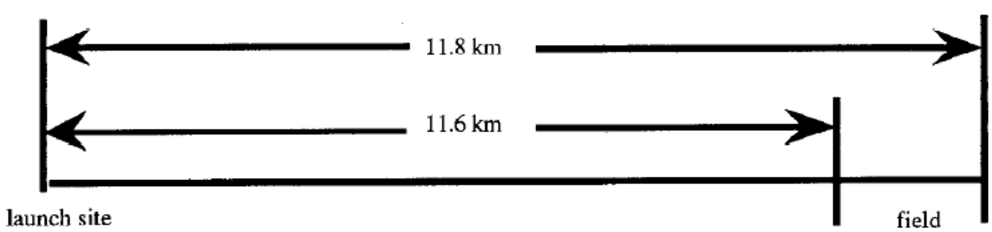
   b. What is Andy's velocity at $t = 1$ minutes and at $t = 2$ minutes?
   c. Write an algebraic expression for the velocity function that models this situation.
   d. Look carefully at the velocity graph sketched in part (4a). Can the total distance Andy traveled be determined from the graph of his velocity function alone? Explain.
   e. How far does Andy travel in these 3 minutes?

5. The horizontal **velocity** of a certain type of rocket is modeled by the following graph. Note that this is a velocity function, **not** a position-versus-time function.



The rocket is launched from a law enforcement compound intended to hit a marijuana field that is surrounded by the archaeological ruins of an ancient village. The village is fortified (though not inhabited) and is of historical importance. Officials are interested in destroying the marijuana field without disturbing the ruins. See the picture below.

Note: Each large square in the graph represents: 1 km/sec × 0.5 sec = 0.5 km



a. Using the graph of the velocity function, how can the distance the rocket will travel be determined? Give a graphical interpretation of distance for this velocity function.

b. From the graph of the velocity function, determine whether the rocket will hit the field, land short of the field (thereby causing damage to the ruins), or land beyond the field (also causing damage).

   Note: You will need to determine both a low estimate for the distance traveled and a high estimate. The boundary of regions used in the estimation cannot overlap the velocity graph.

c. Discuss your results with other members of your class. How can a better approximation of the distance traveled be achieved? Explain.

d. Discuss the problem of how to find the exact distance traveled when given the graph of the velocity function. How does finding overestimates and underestimates for the distance traveled help in determining the exact distance traveled? How can these over- and underestimates be refined?

6. a. Graph $y = 10x^2$ on [0, 2] by [0, 45].[2] Notice that a region under the curve of $f(x) = 10x^2$ on [0,2] is enclosed by the lines $y = 0$ and $x = 2$, and by the graph of $y = f(x)$.

b. i. Without changing the y-interval, graph $y = f(x)$ for x in [1, 2].[3]

   ii. Graph $y = f(1)$ and $y = f(2)$.[4] Sketch the results.

c. Change the domain to [1.5, 2], graph $f(x) = 10x^2$, then sketch $y = f(\text{Xmin})$ and $y = f(\text{Xmax})$ for the x-interval [1.5, 2].

d. Repeat part (6c) for x in [1.9, 2].

e. Repeat part (6c) for x in [1.99, 2].

f. Repeat part (6c) for x in [1.999, 2].

g. As h decreases in size, describe the appearance of the graph of $f(x) = 10x^2$ from $x = 2 - h$, to $x = 2$.

h. Describe how the line $y = f(\text{Xmin})$ approximates $y = f(x)$ on the interval $[2 - h, 2]$ for small h.

i. Describe how the line $y = f(\text{Xmax})$ approximates $y = f(x)$ on the interval $[2 - h, 2]$ for small h.

j. How might this information be useful in determining the distance traveled by an object whose velocity is given by $f(x) = 10x^2$ between $x = 2 - h$ and $x = 2$?

k. Discuss the results obtained in parts (6a) through (6j) with other members of your class before going on to the Section 13.3.

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2 Use program 1 (CASIO 7000), f_1 (CASIO 7700), or Y_1 (TI) to graph this function to save time later.

3 Reset Xmin each time to graph f on the new domain.

4 Program AREA is provided in the appendices to assist in this exploration.

13.3 Numerical Approximations

1. By now you have discovered that the distance traveled by an object whose velocity function f is given graphically can be determined by finding the area under the curve of f. Using a numerical approximation program (with left and right endpoint approximating options) on a computer or calculator, complete the following to approximate the area under the graph of $f(x) = x$ over an interval [a, b], using lower rectangles L_n then upper rectangles U_n, for n subintervals.[5]
   a. Sketch $y = f(x)$ for x in [0, 5].
   b. Suppose $y = f(x)$ is a velocity function. Determine the distance traveled for x in [0, 5] by finding the area under $y = f(x)$ between $x = 0$ and $x = 5$ using a well-known geometric formula.
   c. Using a numerical approximation program on a computer or calculator, determine an estimate of the area under the curve of $f(x) = x$ using rectangles whose height is determined by the function value of the left endpoint of each of 4 equal subintervals. Sketch $y = f(x)$ and the rectangles found. Are these lower or upper rectangles?
   d. Determine an estimate of the area under the curve of $y = f(x)$ using lower rectangles for n = 8, 16, 32, and 64 equal subintervals respectively. Record the results you obtain in the table below.

n subintervals	Area under $y = f(x)$ estimated by lower rectangles, L_n	upper rectangles, U_n
4		
8		
16		
32		
64		

   e. Compare subsequent estimates. As n gets large, is the estimate for the area under the curve improving? Explain.
   f. Repeat parts (1b), (1c), and (1d) using rectangles whose height is determined by the function value of the right endpoint of each of the n subintervals.
   g. Does one numeric method of determining the area under the curve provide a better estimate for the area than another numeric method for this function? Explain.
2. Approximate the area under the curve of $f(x) = 10x^2$ over the interval [0, 2] using lower rectangles L_n then upper rectangles U_n for n subintervals.

[5] Use the program RIEMANN on the CASIO or TI to complete this investigation.

a. Does the sum of the areas of the lower rectangles give an underestimate or an overestimate for the area under the curve of $f(x) = 10x^2$ on the interval [0, 2]? Explain.

b. Do upper rectangles give an underestimate or an overestimate for the area under the curve of $f(x) = 10x^2$? Explain.

c. How can a better approximation for the area under $y = f(x)$ be obtained?

d. List the values of successive underestimates and overestimates in the following table.

n subintervals	Area under $y = f(x)$ estimated by lower rectangles, L_n	upper rectangles, U_n
4		
8		
16		
32		
64		

e. How do the differences between lower and upper approximations for the same number of subintervals compare as the number of subintervals increases?

f. Approximate the distance traveled by an object whose velocity function is $f(x) = 10x^2$ on the interval from 0 to 2.

3. Sketch the graph of $f(x) = \sqrt{4 - x^2}$ for x in [–2, 2].

a. Approximate the area under the graph of $y = f(x)$ for n = 4, 8, 16, 32, and 64 subintervals using lower rectangles and upper rectangles. Complete a table as shown in part (1d).

b. Determine the area under the curve of $y = f(x)$ using a formula from geometry.

c. Does one numeric method of determining the area under the curve provide a better estimate for the area than another numeric method for this function? Explain.