The Logic of Proving Trig Identities

"If I want to add 3 to both sides of a statement I wish to prove is an identity, why can't I?"

This is an excellent question, and it comes down to logic.

Let's look at a "proof" of $\cos^2 x - \sin^2 x = 2\cos^2 x - 1$.

Add 1 to both sides:
$$1 + \cos^2 x - \sin^2 x = 2\cos^2 x$$

Subtract $\cos^2 x$ from both sides:
$$1 - \sin^2 x = \cos^2 x$$

Using the Pythagorean Identity on the left:
$$\cos^2 x = \cos^2 x$$

Therefore, we have proven
$$\cos^2 x - \sin^2 x = 2\cos^2 x - 1.$$

Do you see a problem?

Let's look at the argument being made.

This "proof" says:

- If $\cos^2 x - \sin^2 x = 2\cos^2 x - 1$ is true, then $\cos^2 x = \cos^2 x$ is true.

- $\cos^2 x = \cos^2 x$ is true.

- Therefore, $\cos^2 x - \sin^2 x = 2\cos^2 x - 1$ is true.

Do you see a problem?

Do you see a problem with the following "proof"?

- If I live in Michigan, then I live in the US.

- I live in the US.

- Therefore, I live in Michigan.

You should see that both these arguments are logically invalid.

Let's turn our "proof" of $\cos^2 x - \sin^2 x = 2\cos^2 x - 1$ around.

We know:

$$\cos^2 x = \cos^2 x$$

Apply the Pythagorean Identity on the left:

$$1 - \sin^2 x = \cos^2 x$$

Add $\cos^2 x$ to both sides:

$$1 + \cos^2 x - \sin^2 x = 2\cos^2 x$$

Subtract 1 from both sides:

$$\cos^2 x - \sin^2 x = 2\cos^2 x - 1.$$

Therefore, we have proven

$$\cos^2 x - \sin^2 x = 2\cos^2 x - 1.$$

Do you see a problem?

This "proof" says:

- $\cos^2 x = \cos^2 x$ is true.

- If $\cos^2 x = \cos^2 x$ is true, then $\cos^2 x - \sin^2 x = 2\cos^2 x - 1$ is true.

- Therefore, $\cos^2 x - \sin^2 x = 2\cos^2 x - 1$ is true.

Here, the structure is:

- I like candy.

- If I like candy, then I like lollipops.

- Therefore, I like lollipops.

You should see that our new "proof" is really a proof.

The moral of the story is similar to that of using Law of Sines.

You <u>can</u> use Law of Sines to find angles, but it often causes big problems, so you are better off using Law of Cosines.

You <u>can</u> manipulate both sides of a statement you wish to prove is an identity, but, unless you are well versed in logic, you should not.

When proving an identity:

<u>Pick one side and stick to it!</u>